Studia Humana
Volume 1:1 (2012), pp. 90 — 95.

INTERVIEW
P-ADICS: MATHEMATICS FOR SIGMUND FREUD?

Andrei Khrennikov studied at Moscow State University, Department of Mechanics and Mathematics, in the period 1975–1980. In 1983 he received a PhD in mathematical physics (quantum field theory) from the same department. He started his teaching and research career at Moscow University for Radio-Electronics and Automatics and in 1990 he became full professor at Moscow University for Electronic Engineering. He started his career abroad at Bochum University and since 1997 he is professor of applied mathematics at Linnaueus University, South-East Sweden, since 2002, the director of the multidisciplinary research center at this university, International Center for Mathematical Modeling in Physics, Engineering, Economics, and Cognitive Science. His research interests are multi-disciplnarity, e.g., foundations of quantum physics and quantum information, foundations of probability (in particular, studies on negative probabilities), cognitive modeling, ultrametric (non-Archimedean) mathematics, dynamical systems, infinite-dimensional analysis, quantum-like models in psychology, economics and finances. He is the author of about 400 papers and 16 monographs in journals in mathematics, physics, biology, cognitive science, economics, and finances.

Andrew Schumann: According to Galileo Galilei's famous claim, the book of nature is written in the language of mathematics. Hence, mathematics has been regarded as cornerstone tool in physics since Galilei. His claim is self-evident for physicists till now, but not for philosophers. What do you think how far math can be applied in cognitive sciences? If there are any limits?

Andrei Khrennikov: One of the sources of the extremely successful mathematical formalization of physics was the creation of the adequate mathematical model of physical space, namely, the Cartesian product of real lines. This provides the possibility for "embedding" physical objects into a mathematical space. Coordinates of physical systems are given by points of this space. Rigid physical bodies are represented by geometric figures (cubes, balls, etc.). By describing dynamics of coordinates, e.g., with the aid of differential equations, we can describe dynamics of bodies (from falling stones to Sputniks). For 15 years I have advocated a similar approach to description of mental processes in cognitive sciences and psychology (and even information dynamics in genetics).
Similar to physics, the first step should be an elaboration of a mathematical model of *mental space.* This is a problem of huge complexity and it might take a few hundred years to create an adequate mathematical model of mental space. I recall that it took three hundred years to create a mathematically rigorous model of real physical space. In works many critical arguments have been presented against the real model of space as a possible candidate for a mental space. One of the main arguments was that the real continuum is a continuous infinitely divisible space. Such a picture of space is adequate to physical space (at least in classical physics), but *mental space is not continuous:* mind is not infinitely divisible! Another problem with the real continuum is that it is homogeneous: "all points of this space have equal rights." In opposition to such a homogeneity, mental states have clearly expressed *hierarchical structure.* Therefore a model of mental space that we are looking for should be (at least) *discontinuous* and *hierarchical.* Such models of space were recently invented in theoretical physics. These are non-Archimedean (ultrametric, *p*-adic) physical space.

A.Sch.: What is *p*-adic mathematics and *p*-adic physics? What reasons were to start using non-Archimedean space in physics?

A.Kh.: Special ultrametric number systems, *p*-adic numbers, have been successfully used in quantum physics and string theory (V. S. Vladimirov, I.V. Volovich [10], E. Witten, P. Frampton, G. Parisi, I. Aref'eva. A. Khrennikov [4], and so on). In this approach the main role was played by non-Archimedean features of these systems, violation of Archimedean axiom (which was interpreted as an axiom of measurement theory). The possibility to encode hierarchy by ultrametric was explored in physics of disordered systems (R. Rammal, G. Toulouse, M. A. Virasoro and then G. Parisi, S. Kozyrev, A. Khrennikov [7]), in particular, in physics of proteins (V. Avetisov, S. Kozyrev, etc.), image analysis (J. Benois-Pineau [2], A. Khrennikov, and others), modeling of information processes in complex cognitive and social systems and multivariate data analysis, clustering, data mining (F. Murtagh [8], A. Khrennikov [3]), computer science and cryptography (V. Anashin [1]), bioinformatics.

A.Sch.: What is better in simulating real processes the conventional mathematics or the *p*-adic one? Is reality *p*-adic or conventional?

A.Kh.: In 1994 (in a series of papers in *Theoretical and Mathematical Physics*) two researchers from Steklov Mathematical Institute of the Russian Academy of Sciences started to speculate about a possibility to use *p*-adic numbers in theoretical physics. They speculated that at the fantastically small Planck time and space scales the conventional model of space-time based on real numbers is not applicable. Instead of the real continuum, we have to use *p*-adic space combining discreteness with a new type of continuity – with respect to the corresponding ultrametric. One of important features of this space is violation of the Archimedean axiom which was interpreted by V. Vladimirov and V. Volovich as an axiom on existing of non-commeasurable physical quantities. In 1989 I joined the research group of Vladimirov and Volovich. My interests were in quantum models with wave functions taking values in the fields of *p*-adic numbers and their algebraic extensions, quadratic as well as extensions of higher orders. One of distinguishing features of quantum mechanics with *p*-adic valued wave functions is boundedness of the basic quantum operators, e.g., position and momentum or creation and annihilation operators. As a consequence, Hamiltonians are also bounded. This is valid even for systems with infinite number of degrees of freedom, quantum field theory with *p*-adic valued quantum fields. Another unexpected feature of *p*-adic quantum mechanics is existence of nonequivalent representations of canonical commutation relations in the case of finite number of degrees of freedom.
However, the most intriguing consequence of the usage of *p*-adic numbers for quantization is the appearance of *p*-adic valued probabilities.

A.Sch.: What is non-Kolmogorovian probability theory you are dealing with and how does it connect to *p*-adic mathematics and physics?

A.Kh.: To solve the interpretational problems of *p*-adic quantum mechanics (in the model with *p*-adic valued wave functions), there are defined rigorously *p*-adic probabilities by extending von Mises approach to the *p*-adic case. The starting point is the evident fact that experimental data, including probabilistic data, are always rational. As a consequence of finite precision of any measurement and finite time which can be used to collect data, only rational numbers can be produced by experimenters. In particular, relative frequencies of realizations of events are always rational numbers. On the field of rational numbers von Mises considered the topology which is induced from the field of real numbers. Probabilities were defined as the limits (if they exist) of sequences of relative frequencies. This simple definition (the principle of statistical stabilization of relative frequencies) was combined with rather contradictory definition of a random sequence, collective.
Formalization of the notion of randomness attracted a lot of interests in communities of probability theory and logic. In particular, Kolmogorov proposed theory of algorithmic complexity. I proposed to consider one of *p*-adic topologies on the field of rational numbers containing all relative

frequencies of realization of events. Probabilities were defined as the limits (if they exist) of sequences of relative frequencies. It can happen that, for one prime *p*, the limit and hence probability exists, but for another not; it can happen that it does not exist in the ordinary meaning, i.e., with respect to the real topology, but exists for one of *p*-adic topologies. Hence, the absence of probabilistic regularities in the ordinary sense does not imply that there are no probabilistic regularities, *p*-adic probability may exist. Notice that *p*-adic probability theory is used now in *p*-adic quantum physics and biological modeling (see [5, 6]).
One of interesting applications of *p*-adic probability is a possibility to justify usage of "negative probabilities." Such "probabilities" appear regular in quantum physics. Dirac actively used such "probabilities" to quantize (relativistic) the electromagnetic field. Feynman also applied negative probabilities to quantum foundations. Some authors noticed that Bell's inequality (the fundamental test of compatibility of local realism and quantum formalism) can be violated in classical physical models, but under the assumption that hidden variables can have negative probability distributions. It is impossible to justify usage of negative probabilities in the classical (Kolmogorovian) probabilistic model. To provide a frequency interpretation (which is only useful for practice) is especially difficult.
In the framework of *p*-adic probability theory negative (rational) probabilities were defined in the mathematically rigorous way. Negative rational numbers (as well as all rational numbers) can be embedded in any field of *p*-adic numbers. If a sequence of relative frequencies for trials for some random event converges in the *p*-adic topology (for some prime number *p*) to a negative rational number, this number is by definition the negative probability of this random event. The same negative rational number can be embedded in various fields of *p*-adic numbers. Its probabilistic meaning depends on the prime number *p* and topology of statistical stabilization.

A.Sch.: Have you ever met Andrei Nikolaevich Kolmogorov, the founder of axiomatic probability theory?

A.Kh.: Yes. First of all, I met him regular as a student during the course on mathematical logic which he gave to us, students of the Dept. of Mechanics and Mathematics of Moscow State University. At one occasion (the submission of a note to "*Doklady of Academy of Science of USSR*", DAN), see [9], I, the first year graduate student, was introduced (by my supervisor, Prof. Smolyanov) to Andrei Nikolaevich Kolmogorov who at that time was physically very disable, but mentally bright. The paper (joint with Prof. Smolyanov) was devoted to a generalization of probability theory to complex valued probabilities. It attracted interest of Kolmogorov. In the discussion on the main ideas of this paper he stressed the role of the frequency definition of probability by von Mises and related logical problems. This conversation played a crucial role in forming of my interest to foundations of probability theory, especially the frequency definition.
Andrei Nikolaevich asked for a copy of the note for his personal use. At that time the only possibility to make a copy was to tape the paper once again – the use of copy-machines was under the strong control. It is a pity, but the graduate student Andrei Khrennikov was so busy with his own "very important tasks" that Andrei Nikolaevich had never got a possibility to study this paper in details. The Editors of DAN sent the note recommended by Kolmogorv to a reviewer who wrote a negative report with motivation that the axiomatics of probability theory had been established in 1933 and there are no reasons to try to generalize this axiomatics. Of course, the editors did not accept such a report written on the paper recommended to publication by the creator of the axiomatic of probability theory. So, finally the paper was published only three years later. At that time Kolmogorov was already very disable physically and only his former students had possibilities to visit him. Albert Nikolaevich Shyryaev told that Kolmogorov was still mentally active; Shyryaev read him whole days mathematical books.

A.Sch.: How ultrametric can be used in cognitive science?

A.Kh.: It should be stressed here that actually the *new approach based on ultrametricity enables one effectively use "continuous" methods to study "discrete" problems*, both of theoretical and applied origin. This is the core of leading scientific idea of my research.

The simplest class of ultrametric spaces is given by homogeneous *p*-adic trees (here *p* is a prime number giving the number of branches of a tree at each vertex). It is interesting that such trees are nicely equipped: there is a well-defined algebraic structure which gives the possibility to add, subtract, multiply, even divide branches of such a tree. There is a natural topology on such trees encoding the hierarchic tree structure. This topology is based on a metric, so called *ultrametric.* Thus *p*-adic trees are not worse equipped than the real line. However, the equipment – algebra and topology – is very different from the real one. I proposed to choose *p*-adic trees as possible models of *mental space*: points of this space – *mental points* – are branches of a tree. It is possible to encode tree's branches by sequences of numbers. These are *mental coordinates* representing *mental points.* By using mental coordinates we are able to embed into the space mental analogs of physical rigid bodies – *basic categories (special associative classes) and ideas.* They are represented, respectively, by *balls and collections of balls* in the ultrametric mental space. Association-relation (which is an equivalence relation for mental points) is based on ultrametric. The use of ultrametric is crucial! In what following we call basic categories simply by categories. But we emphasize that these are specific categories coupled to mental ultrametric.

Mental points (represented by branches) are the *elementary mental entities.* A category is represented as a subset of the mental space. The crucial point is that the associative coupling of mental points is fundamentally hierarchical. Therefore a category is not an arbitrary set of mental points, but a hierarchically coupled collection.

Since in our model the mental hierarchy is encoded by the topology of the mental space, it represents the associative coupling of mental points into balls. A larger ball couples together more mental points. Thus it is a more general category. Decreasing of a ball's radius induces decreasing of generality of a category which is represented by this ball. It becomes sharper. In the limit, we obtain the ball of zero radius. That is nothing else than a single mental point (the center of such a degenerated ball). This is the limiting case of a category. We remark that the real brain produces *finite mental trees.* For such a finite tree, each point (its branch) is simply a ball of finite radius (it is determined by the size of the tree). It is a trivial associative-class: the mental point associated with itself. However, consideration of idealized mental spaces based on infinite trees is an extremely useful mathematical abstraction – as well as consideration of continuous real line or plane. In principle, even in physics one can work on e.g. discrete plane. However, the real analysis is developed for its continuous idealization. Therefore it is convenient to work on the real "continuous plane." In the same way a powerful ultrametric analysis was developed for a (special) class of infinite trees and it is convenient to use such spaces for mathematical modeling, e.g., in psychology or cognitive science.

I applied such an approach to mathematical modeling of Freud's theory of interaction between unconscious and conscious domains. One of basic features of my model is splitting the process of thinking into two separate (but at the same time closely connected) domains: *conscious* and *unconscious*. I use the following point of view on the simultaneous work of the consciousness and unconsciousness. The consciousness contains a *control center CC* that has functions of control over results of functioning of subconsciousness. *CC* formulates problems, and sends them to the unconscious domain. The process of finding a solution is hidden in the unconscious domain. In the unconscious domain there work complex dynamical systems – *thinking processors.* Each processor is determined by a function *f* from mental space into itself (describing the corresponding feedback process – psychological function). It produces iterations of points of mental space. These intermediate mental points are not used by the consciousness. The consciousness (namely *CC*) controls only some exceptional moments in the work of the dynamical system in the unconscious domain – attractors and cycles. Dynamics of mental points induce dynamics of mental figures, in particular, ball-categories and, hence, ideas (collections of balls). The crucial point is that behaviors of the dynamical in the mental space and its lifting to spaces of categories and ideas can be very different. Extremely cycling (chaotic) behavior on the level of mental points (and even categories) can imply nice stabilization to attractors on the level of ideas. Therefore it is profitable for the brain (modeled in this framework) to use as solutions of problems attractors on the level of ideas and not mental points (and categories).

The computational ("thinking") machine described in my works represents an *unconventional computation in ultrametric mental space.* I notice that the statement that Turing machines completely express the intuitive notion of computing is a common misinterpretation of the Church-Turing thesis. For instance, Turing asserted that Turing machines could not provide a complete model for all forms of computation, but only for algorithms. Therefore he defined choice machines as an alternative model of computation, which added interactive choice as a form of computation, and later, he also defined unorganized machines as another alternative that modeled the brain.

A.Sch.: Are you the first who proposed to use math in explicating conscious and unconscious ideas?

A.Kh.: One of the first scientists who proposed a mathematical simulation of processing mental information was the famous 19th-century philosopher Johann Friedrich Herbart (1776 – 1841). His model was based on real analysis and classical mechanics (e.g., see his *Psychologie als Wissenschaft* written in 1824 – 1825) and it assumed the difference between conscientious and unconscious ideas too.

A.Sch.: Where can your *p*-adic cognitive science be applied?

A.Kh.: In series of models I considered cognitive systems with increasing complexity of psychological behavior determined by structure of flows of categories and ideas. Using this basic conceptual repertoire an increasingly refined cognitive model is developed starting from an animal like individual whose sexual behavior is based on instincts alone.

At the first step a classification of ideas to interesting and less interesting ones is introduced, and less interesting ideas are deleted. At the next level a censorship of dangerous ideas is introduced and the conflict between interesting and dangerous leads to neurotic behaviors, idees fixes, and hysteria. These aspects of the model reflect more the general structure of conscious/unconscious processing rather than properties of *m*-adic numbers. I stress again that the basic mathematical structure for this model is mental ultrametric space. In particular, ultrametric is used to classify ideas – to assign to each idea its measures of interest and interdiction.

One of my interests is creation of *psycho-robots*, exhibiting important elements of human psyche, including the presence of two blocks: unconscious and conscious. Creation of such psycho-robots may be useful improvement of domestic robots. At the moment domestic robots are merely simple working devices (e.g. vacuum cleaners or lawn mowers). However, in future one can expect demand in systems which be able not only perform simple work tasks, but would have elements of human self-developing psyche. Such *AI*-psyche could play an important role both in relations between psycho-robots and their owners as well as between psycho-robots themselves. Since the presence of a huge numbers of psycho-complexes is an essential characteristic of human psychology, it would be interesting to model them in the *AI*-framework.

References

1. Anashin V., "Non-Archimedean ergodic theory and pseudorandom generators", *The Computer Journal* (Oxford), 53(4), 2010, p. 370—392.
2. Benois-Pineau, J., and Khrennikov A. Yu., "Significance Delta Reasoning with p-Adic Neural Networks: Application to Shot Change Detection in Video", *The Computer Journal* (Oxford) 53(4), 2010, 417—431.
3. Khrennikov A., *Information Dynamics in Cognitive, Psychological and Social Phenomena (Fundamental Theories of Physics)*, Kluwer, Dordrecht, 2004.

4. Khrennikov A. Yu., *P-adic Valued Distributions in Mathematical Physics*, Kluwer, Dordrecht, 1994.
5. Khrennikov A. Yu., *Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models*. Kluwer, Dordreht, 1997.
6. Khrennikov A. Yu, *Interpretations of Probability*. De Gruyter, Berlin, 2009 (second edition).
7. Khrennikov A. Yu., Kozyrev, S.V., "Replica symmetry breaking related to a general ultrametric space I—III" *Physica A: Statistical Mechanics and its Applications* 359, 2005, 222—240; 359, 2006, 241—266; 378, 2007, 283—298.
8. Murtagh F., "On ultrametricity, data coding, and computation", *J. Classification*, 21, 2004, 167—184.
9. Smolaynov O. G., Khrennikov A. Yu., "Central limit theorem for generalized measures on infinite-dimensional space". *Doklady Akademii Nauk USSR*, 281 (2), 1983, 279—283 (in Russian).
10. Vladimirov V.S., Volovich V.S., Zelenov, E.I., *P-adic analysis and mathematical physics*. WSP, Singapore, 1995.