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Many-Valued Logic in the Jewish Short Stories

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Abstract:
Jewish short stories (parables, tales, jokes, etc.) are explained from the viewpoint of many-valued logic. On the basis of some examples, we show, how their contents may be logically interpreted.

Keywords: Jewish study, two-valued logic, many-valued logic.

The defect in the knowledge of mathematic
evolves the hundredfold defect in the Tora's knowledge
The Gaon

1. Introduction

The problem of the national identification is not new. People may be considered one nation on the basis of common in behavior: common religion, common language, common traditions, connected with the same motherland. All this approaches have their own merits and demerits. For example, when we use religious identification we can call the Jews only those who follow Judaism; when we use ethnical identification – only those who speak Hebrew and knows the Jewish culture and traditions; when we use commune identification – only those who belong to the Jewish community; when we use the state identification – only those who lives in Israel and so on. We propose a new method to national self-identification of Jews: people belong to the same nation on the basis of logical ways of creating their short stories about themselves: parables, tales and so on. In particular, we consider ways of creating Jewish short stories by the implicit appealing to many-valued logic. This feature of short stories concerns intellectual activities and behaviors and can serve as one of the indices for Jewish cultural identification in the limits of the subject.

2. Two-valued and Many-valued Logic

Every individual uses one or another type of logic in his creative work (usually by intuition). The logic divides on two-valued and many-valued which have essential differences. These differences are as follows:

1. For two-valued logic any statement can have only two truth values: to be true or false, and the negation of one side gives another side. For many-valued logic, any statement can have some

truth values beginning from three (true, false, indeterminate) values and the negation of given truth value can give us any other truth value in accordance with defining the negation operation.

2. For two-valued logic, it is possible to return to any statement by the way of its double negation (the law of double negation). For many valued logic, this returning is always not possible and when it is possible it is controlled by more complex laws (the law of the triple negation and so on).

3. For two-valued logic, if the given statement is true or false, the third possibility is excluded (the law of excluded middle). For many-valued logic, the choice is more complex: in one case the given statement is true, in another – its negation is true, in the third – another statement which differs from the previous is true. The concrete rule of choice depends on the negation operation definition. So, for many-valued logic the choice is more complex: in one case the given statement is true, in another its negation is true. Hence, for many-valued logic the choice of true statement from the pair (affirmation and negation) in contrast to two-valued logic is impossible.

4. For two-valued logic, the given logical statement and its negation cannot be true or false simultaneously: when one is true, another is false and vice versa. So, in two-valued logic the contradiction law holds. For many-valued logic, the simultaneous truth of two opposite statements is not eliminated. So, many-valued logic does not declare the negation of some affirmation false when the affirmation itself is true and true when it is false in contradiction to two-valued logical reasoning by this simple scheme.

5. In two-valued logic, we can transfer the negation operation over a complex statement obtained by using of the two connectives OR, AND if we change OR by AND and AND by OR (the law of de Morgan). In other words, we can replace the statement of the form NOT(*A* OR *B*) by (NOT *A*) AND (NOT *B*), the statement NOT(*A* AND *B*) by (NOT *A*) OR (NOT *B*). For many-valued logic, such replacing is always impossible and if it is possible, it is subordinated to more complex laws, depending on the negation operation.

The difference between two-valued and many-valued logics can be represented visually by colors: the value to be true may be presented by white, the value to be false may be presented by black and other truth values by other colors. Then we can say that for two-valued logic there exists only the black-white world with simple transition from one color to another with by using the negation operation (not-white is black, not-black is white). For many-valued logic, the world is many-colored, the transition from one color to another with the help of logic can fail and if it is possible, it needs more complex operations.

3. The Basic Hypothesis

The Jewish short stories have logical forms described by many-valued logic. It does not mean that all the Jews are many-valued thinkers and all Gentiles are two-valued: both have thinkers of different ways. However, any interaction of individuals in telling short stories about self-identities with a similar way of thinking causes differences in the group behavior. It allows us to identify the Jews with their characteristic tradition of creating short stories about themselves. The essence of the problem of multi-cultural communications is that for two-valued thinking many-valued thinking is not so understandable, because it contents unusual ideas such as a lack of two truth values (true or false) or two evaluations of acts (good and evil) as well as an impossibility of simple transitions from one value or evaluation to another by the rules of such type: not-true is false, not-good is evil and so on.

4. Some Typical Examples

Some examples given below illustrate the proposed hypothesis and demonstrate how many-valued logic is used in the Jewish short stories. These examples were collected by me among short stories which are popular among the Jews in Russia.

1. Two Jews asked their Rabbi. One of them said: “He borrowed money from me and he did not return his debt. Let him return his debt”. And the Rabbi answered: “You are right”. Another Jew said: “I promised to return my debt when I'll earn money, but I have no work and have no money yet”. The Rabbi said: “You are right, too”. But the Rabbi's wife asked: “How both of them can be right? One required to return his money and another refused to do it”. “You are right, too”, said the Rabbi.

2. The patriot says: “I cannot live a day out of my motherland. I understand Pushkin well. Do you remember, he calls the wish of changing the residence place the evil voice?” “Have a pity!” the Jew objects. “Why do you sit in your town if even it is very good? There are so many wonderful cities and every one of them will help me to get to know the wide world, myself and what is good in my town. So the wish for travelling is not the evil voice, but it is a real God's gift”.

3. To make money or to build a just society? The merchant says: “I choose the first. I am interested in money, not in justice”. The socialist says: “I choose the second, because only in the just society one can earn money honestly”. “My choice is such”, says the Jew-billionaire, “To make much money and endow part of it for building the just society. I consider that one does not contradict to another: making money helps me to establish the justice and the just society helps me to make money honestly”.

4. To love myself or others? The egoist says: “I choose the first variant of behavior”. And the altruist says: “I choose the second”. But the wise man Hillel taught: “If I am not for myself, then who is for me? But if I am only for myself, then what for I am?”

5. Once the gentile asked the Jew: “Why even if you finish successfully your business, don't you enjoy yourself very much and even if you lost the game, don't you grieve too much?” And he got from the Jew such an answer: “No one success can be full – there are some elements of the possible loss in it, because of that we do not enjoy too much! And any loss cannot be full – it is always – has the elements of the possible winning in it, because of that we do not grieve too much”.

6. “Why have the Jews studied so many commentaries in their holy books for centuries though these commentaries content many contradictions?” asked the gentile. The Rabbi answered that this commentaries were written by wise men so only keeping all commentaries even contradictions, we get the truth. We do not throw out the splinters of the broken diamond only because on one of them is written “Yes” and on others “No”. All splinters are parts of one beautiful diamond.

7. All thinking people are parted on the people of faith and the people of mind. “And who are you, the Jews?” – once the gentile asked. And the Jew answered: “We are people of law. We do not crave for opening the world laws as the people of mind do and we do not crave to understand the wonder of the world laws as the people of faith do. Ideal world laws exist in the form of Torah and our problem is only to set up the conformity between the real world and Torah's laws and to expand the field of this accordance”.

A huge number of Jewish short stories were collected and discussed in books [1], [2], [4], [5]. Notably, one of the first scholars, who started to philosophically analyze the Jewish short stories, was Sigmund Freud [3].

5. Conclusion

On the basis of hypothesis about many-valued logical interpretation of some Jewish short stories gives us possibility to understand the character of thinking of the Jews and their self-perceptions.

References

1. J. Allen. *500 Great Jewish Jokes.* Signet, 1990.
2. A. Berger. *The Genius of the Jewish Joke.* Jason Aronson, 1997.
3. S. Freud. *Der Witz und seine Beziehung zum Unbewußten*. 1905.

4. E. Oring. *The Jokes of Sigmund Freud.* University of Pennsylvania Press, 1984.
5. J. Telushkin. *Jewish Humour: What the Best Jewish Jokes Say About the Jews.* Harper Paperbacks, 1998.

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The Downward Causality and the Hard Problem of Consciousness or Why Computer Programs Do not Work in the Dark

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Abstract:
Any low-level processes, the sequence of chemical interactions in a living cell, muscle cellular activity, processor commands or neuron interaction, is possible only if there is a downward causality, only due to uniting and controlling power of the highest level. Therefore, there is no special “hard problem of consciousness”, i.e. the problem of relation of ostensibly purely biological materiality and non-causal mentality – we have only the single philosophical problem of relation between the upward and downward causalities, the problem of interrelation between hierarchic levels of existence. It is necessary to conclude that the problem of determinacy of chemical processes by the biological ones and the problem of neuron interactions caused by consciousness are of one nature and must have one solution.

Keywords: consciousness, hard problem, downward causality.

The hard problem of consciousness was formulated by David Chalmers as a necessity to complement human brain functioning with a conscious experience and subjective mental pictures. (“The really hard problem of consciousness is the problem of experience” [1, p. 201]). This is definitely a stumbling rock for any theories of consciousness, based on the conception of causal closure of the physical: if brain can process all information, transform all the incoming signals into actions without any subjective feelings, then for what brain needs consciousness at all? Let us try to analyze the problem, using the example of computer analogy, i.e. let us focus our attention on a classical Chalmers’ question: “why doesn't all this information-processing go on ‘in the dark’?” [1, p. 203], in relation to a computer.

At the first glance, this question, when asked about computer functioning, seems just trivial and good-for-nothing. Any “dummy” can understand that while, for example, converting a video file we can turn off the screen, PC-speakers, printer and after this a computer will successfully complete an information processing “in the dark” paying no attention to our actions. And at this point we can say that images on the screen (for example, a line which shows us the status of a process) are nothing more than “epiphenomena” – only the “accompaniment” (in Chalmers’ terms) of a real information processing. There seems to be a complete analogy between the information processes in our brain and the mental picture in our consciousness on one hand and the information processing and its display on the screen on the other hand.

But something compels us to resist the temptation of such an easy solution. There is something wrong with this analogy. We understand that a computer, which works only for itself, is a senseless thing, a thing-in-itself. We realize that everything we can interpret as an external “mental” manifestations of computer functioning: a video edit, a photo correction, a computer game, etc. – all this makes computer useful. In other words, the picture on the screen is not an "accompaniment" to executed code, but this code has sense only when it is mentally determined, when it performs a certain "higher" function.

And this conclusion about the priority of "the mental", "the higher" in computer over its "lower" information processing becomes more obvious, if we focus our attention at programming. Let us ask the question: from where comes the lowest code level? It is clear, that it is created by a programmer specially and purposefully for providing the "higher" functions. And after that the original question gains the new sense: why a processor does not work "in the dark"? Or, to be more precise, it becomes obvious, that the question is senseless – if there was no original "mental" light, if the "higher" sense of program's work was not formulated, there would be no program.

Of course, the program itself sometimes can work "in the dark". Moreover, there is no necessity to bring to light (to display on the screen) all cycles of array sorting and database queries. Of course, a program is largely autonomous – it does not need a picture on the screen, it does not need to know, what data it works with to perform its function – it needs only some instructions – to take a byte from here, to add one bit, to put it there. But without an external manifestation, without its "higher" sense a program could not come to light, or, to be more precise to flash itself in the "darkness" of computer memory.

After this computer analogy is explicated Chalmers' question ("why doesn't all this information-processing go on 'in the dark'?") asked with the reference to consciousness seems not as sensible as before. Here one is tempted to clarify a point by asking: from where do these "information processes" come in this darkness. Of course, if we are interested only in reactions of a zombie-machine to external signals, the description at the level of "lower informational processes" will surely be enough for us. However, if we face the problem of explaining the nature of reasonable thinking, ethic and aesthetic perception, how can we ask such a stupid question: why all this is in the light? That is because in the dark, at the physiological level there is nothing for which, in fact, they are meant. As we cannot see a text or a picture in a program code, we cannot find any thoughts or emotions at the level of neuron interaction. So we can consider as more sensible the thesis that the lowest procedural level only provides the existence of a mental reality, which is goal-setting for it. The functional requirement list makes the activity of a programmer sensible in the same way. In other words, neither a computer, nor brain can exist without a "mental light" – the "light" precedes a "hardware" and a "hardware" is created for it. Here we deal with a downward causality, not with a linear one.

Naturally, some doubt arises as to the adequacy of given analogy: a computer has no consciousness. But here it is noteworthy that the only difference between a man and a computer is that in the first one (in a man) the mental and the procedural are combined in one "device" and in the second one (in a computer) the "dark" and the "light" levels are distributed in time and space. The whole sense of the computer analogy is that the lowest level of a computer, which seems to be able to work "in the dark", not only can't do it, but it would not appear at all without a "light" level of programmers and users. In other words, the mental levels of both a man and a computer are necessary and fundamental. They form the lowest procedural level and make it sensible. In other words, the problem is not whether a computer has consciousness or not, but in the fact that no "dark" procedurality is possible and sensible without a mental level ("inner" or "external" one).

At this point, we face the main problem – from where comes at the mental level the "requirement list", according to which programs of "information processes" are created in the head of a man? The problem is interesting and really hard. However, in spite of its difficulty, it is far more sensible than a search for aesthetic emotions at the level of neuron interaction. The answer to this question really promises the productive solutions of consciousness problems, unlike the attempts to reveal how the philosophical ideas "look" at the level of metabolite interchange between neurons.

But let us return to the "hard problem of consciousness" and try to reformulate another Chalmers' question: "why is the performance of these functions [perceptual discrimination, categorization, internal access, verbal report] accompanied by experience?" [1, p. 204]. Let us say it like this: why chemical reactions are accompanied by mitosis? Or at the physiological and behavioral level: why running of a cheetah is accompanied by the mental action of a "pursuit" and running of a fallow deer is accompanied by an action of "escaping"? We can formulate this question in relation to a computer: why the accomplishment of a command sequence is accompanied by firing of the game characters on the screen?

And at this point, the absurdity of original question becomes obvious. Chalmers' question tacitly implies that the listed low-level processes (chemical reactions, muscle actions, work of a processor, neuron interaction) *can proceed per se, independently of higher properties, allegedly only accompanying.* But it is clear, that there is no such purely chemical reaction as "mitosis" – it is just stupid to say that a chemical process is accompanied by mitosis. There can be no just running – it is always mentally motivated and determined by a higher aim in relation to physiology: its beginning and end is not determined at the cellular level of muscles. It is senseless to speak about a program code, which is not written for realization of certain high-level tasks – video processing or spell checking.

In actual fact, it is impossible to explain mitosis, hunt of a cheetah, changing pictures on the screen only as *interpretations, as epiphenomena of self-sufficient, autonomous low-level processes.* The opposite is obvious: the sequences of elementary chemical reactions, muscle actions and processor commands are determined by these very epiphenomena – biological life of a cell, psychic activity of higher Metazoans, the will of a programmer. Literally, every next chemical reaction in mitosis is not determined by a previous reaction. All chemical reaction together implement the mitosis, but these elementary interactions do not comprise a single chemical process, united self-sufficient chemical reaction – here we have *many simultaneous processes united by a non-chemical causality.* Just like this a muscle contraction of a cheetah's hind leg is not determined by its previous contractions – *we cannot speak about any causal closure at physiological level, for which psychic is only an external optional form.* We have just an opposite picture – all physiological processes are subject to the highest psychic goal-setting.

Here the question suggests itself: why when formulating "the hard problem of consciousness" the neuron substrate of human brains is placed in the causally closed physical world? Chalmers puts the problem of casual dependence of consciousness and neurophysiologic processes as if the problem of determination of life, its reduction to chemical processes and further to physical interaction has already been solved. After all, such "hard problem" can also be formulated with respect to interrelation of biologic metabolism and organic synthesis reactions: why the causally closed chemical processes are accompanied by the "biological experience"? In other words, if it is acceptable to speak about the causal closure of the physical, then only at chemical level and lower levels – and according to the "hard problem" logic, life should be interpreted only as an epiphenomenon.

Why at least some hints of solution of consciousness problem along the lines of epiphenomenalism or psychophysical parallelism are possible at all? Problems of relation between low-level and high-level properties, which were already formulated for relation between the chemical and the biologic, the physiologic and the psychic, definitely indicate the ontological inadequacy of such approach. Where can we see the difference between the relation, on the one hand, of muscle cells and psyche, and, on the other hand, of neurons and consciousness? Why it is so obvious to us that the state of one kind of cells (muscle cells) is determined by the high-level causality – the mental actions like "pursuit" or "escaping" – and the states of another kind of cells (neurons) determine each other like gears in a clock? Why do we understand that life is something more than a chemical reaction, and at the same time accept that the consciousness is only a name for a self-sufficient sequence of neuron interaction? Of course, even running action is a sequence of cellular activity, but the *problem is not in realization, it is in causality.*

So, the simple analysis of empiric facts makes us come to the following conclusion: any low-level procedurality – the sequence of chemical interactions in a living cell, muscle cellular activity, processor commands or neuron interaction *is possible only if there is a downward causality, only due to uniting and controlling power of the highest level* (see [3, pp. 86–101]).

Therefore, there is no special "hard problem of consciousness", i.e. the problem of relation of ostensibly purely biological materiality and non-causal mentality – we have only the single philosophical problem of relation between the upward and downward causalities, the problem of interrelation between hierarchic levels of existence. It is necessary to conclude, that the problem of determinancy of chemical processes by the biological ones and the problem of neuron interactions caused by consciousness are of one nature and must have one solution.

References

1. Chalmers, D.J. Facing up to the problem of consciousness, *Journal of Consciousness Studies*, 2 (3), 1995, pp. 200–219.
2. Chalmers D.J. *The Conscious Mind. In Search of a Fundamental Theory*. N. Y.: Oxford, 1996.
3. Boldachev, A.V. *Temporality and the philosophy of absolute relativism*, Moscow: Lenand, 2011 (in Russian).

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Towards New Probabilistic Assumptions in Business Intelligence

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Abstract:
One of the main assumptions of mathematical tools in science is represented by the idea of measurability and additivity of reality. For discovering the physical universe additive measures such as mass, force, energy, temperature, etc. are used. Economics and conventional business intelligence try to continue this empiricist tradition and in statistical and econometric tools they appeal only to the measurable aspects of reality. However, a lot of important variables of economic systems cannot be observable and additive in principle. These variables can be called symbolic values or symbolic meanings and studied within symbolic interactionism, the theory developed since George Herbert Mead and Herbert Blumer. In statistical and econometric tools of business intelligence we accept only phenomena with causal connections measured by additive measures. In the paper we show that in the social world we deal with symbolic interactions which can be studied by non-additive labels (symbolic meanings or symbolic values). For accepting the variety of such phenomena we should avoid additivity of basic labels and construct a new probabilistic method in business intelligence based on non-Archimedean probabilities.

Keywords: business intelligence, additivity, measurability, non-additive measures, symbolic interactionism, symbolic value, non-Archimedean probabilities.

1. Symbolic Values as Non-Additive Measures

In business intelligence the majority of expert systems used to analyze an organization's raw data appeal to appropriate statistical and econoementric tools [2], [10], [11], [16]. In their possible applications they are extremely limited by some fundamental assumptions about the characters of material laws. First of all it is assumed that the system of the material universe consists of primary bodies (atoms) and their combinations and relationships described by mathematical equalities, in particular it is supposed that each atom bears its own separate and independent effect so that the total state is being compounded of a number of separate effects detected in the proceeding state. In other words, in order to explore the total state we should present an appropriate proceeding state as a machine:

> And in this matter the example of several bodies made by art was of great service to me: for I recognize no difference between these and natural bodies beyond this, that the effects of machines depend for the most part on the agency of certain instruments, which, as they must bear some proportion to the hands of those who make them, are always so large that their figures and motions can be seen; in place of which, the effects of natural bodies almost always depend upon certain organs so minute as to escape our senses. And it is certain that all the rules of mechanics belong also to physics, of which it is a part or species, [so that all that is artificial is withal natural]: for it is not less natural for a clock, made of the requisite number of wheels, to mark the hours, than for a tree, which has sprung from this or that seed, to produce the fruit peculiar to it. Accordingly, just as those who are familiar with automata, when they are informed of the use of a machine, and see some of its parts, easily infer from these the way in which the others, that are not seen by them, are made; so from considering the sensible effects and parts of natural bodies, I have essayed to determine the character of their causes and insensible parts (René Descartes, *Principles of Philosophy*, 1644; translated by John Veith).

René Descartes was one of the first thinkers who have put forward the assumption that wholes can be studied due to laws of connection between their individual parts described my maths, i.e. wholes are subject to different laws in proportion to the differences of their parts and these proportions can be analyzed mathematically. This one of the main presuppositions of mathematical tools in science is called *measurability* and *additivity* of reality. Due to this assumption modern physics can have obtained all its results. For discovering the material universe it has appealed to *additive measures* such as mass, force, energy, temperature, etc. Economics and conventional business intelligence try to continue this empiricist tradition and in statistical and econometric tools they deal only with the measurable aspects of reality. They try to obtain additive measures in economics and in studies of real intelligent behavior, also.

Nevertheless, there is always the possibility that there are important variables of economic systems which are unobservable and non-additive in principle. We should understand that statistical and econometric methods can be rigorously applied in economics just after the presupposition that the phenomena of our social world are ruled by stable causal relations between variables. However, let us assume that we have obtained a fixed parameter model with values estimated in specific spatio-temporal contexts. Can it be exportable to totally different contexts? Are real social systems governed by stable causal mechanisms with atomistic and additive features?

In the 19th century there was a causal relation between power demand and good and service consumption: the increase of good and service consumption has implied the increase of power demand. But now this relation is untrue, because power demand does not increase and good consumption does. Hence, the same causal relation was true in the industrial society and false in the post-industrial society. In other words, that fact shows that in real social systems there is *no ergodicity*. Recall that in case of ergodicity we can describe a dynamical system which has the same behavior averaged over time as averaged over the space for all states. Therefore it is sophisticated to find out additive measures in economics at all.

One of the additive measures that have been widely applied in economies is *money*. Due to money we can compare goods and services as well as capitals. *Economic capital* is the term to describe already-produced goods or any asset that is used in production of goods or services. There is also its part, *financial capital*, to denote money used to buy what is needed to provide services to the sector of the economy upon which an appropriate operation is based. Money allows us to evaluate material welfare, goods, and services. Nevertheless, we can face non-additivity there too. The matter is that some welfare is not additive. For example, two oil-paintings with the same parameters can have so different surplus exchange values: they can become cheap, expansive or precious.

Goods and services have a high dollar surplus exchange value if they are produced as a part of *symbolic capital* [5], [6] which denotes a non-economic capital (such as education, networking, power, publicity, image) allowing us to aid social exchange. Economic capital consists of any resources which can be used for producing goods or services to obtain profit. Symbolic capital consists of cultural values of goods and services which increase their surplus exchange values extremely.

The Karl Marx's economic theory [1], [9] tried to describe causal connections of industrial society that was concentrated on producing goods. But the modern society is post-industrial and it is concentrated on producing services, where symbolic capital plays more significant role than it took place in the industrial society. According to Marx, any society has the following two levels: (i) the base (relations of production, relations of production forces) and (ii) superstructure (cultural, symbolic relations). The superstructure is derivable from the base. In the industrial society there were not enough places for symbolic capital. The transition from the feudal formation to the capitalist one is, first of all, a reduction of symbolic capital, its depreciation. Public statuses, titles of noble families were not as important as the economic capitals.

The role of symbolic capital has mainly increased in the post-industrial society. It is caused by a priority which services have over goods now in earning money and obtaining profits. In services there is always an appreciable share of symbolic capital and symbolic values. In sausages or tooth-brushes there is no *symbolic values* (as well as in other consumer goods), but if we take fashion shows or cinema there is already nothing more than symbolic values. Accordingly, surplus values can be so different. In the modern society the Marx's scheme about the base domination over the superstructure is not true. Nowadays the superstructure already determines the base. Symbolic capital dominates over economic capital. Any development of information technologies only strengthens this domination. Money and goods are connected now with social exchanges mediated by information technologies. Such a revaluation began to transform promptly all societies towards increasing the importance of publicity and openness. Any society with the higher role of symbolic capital becomes transparent.

Symbolic values which are involved now in producing goods and services cannot be additive measures. However, they can be studied within s*ymbolic interactionism*, the theory developed since George Herbert Mead [12], [13], [14] and Herbert Blumer [3], [4]. They have stated that people act toward things based on *symbolic meanings* they ascribe to those things. In turn, these meanings are derived from social interactions and transformed through their interpretations. Symbolic meanings are defined and studied by qualitative research methods.

Thus, in statistical and econometric tools of business intelligence we accept only phenomena with causal connections measured by additive measures. Nevertheless, in the social world we deal with symbolic interactions studied by *non-additive labels* (symbolic meanings or symbolic values). For accepting the variety of such phenomena we should avoid additivity of basic labels.

2. Basic Assumptions of Probability Theory and Non-Additive Non-Archimedean Probabilities

Since Descartes and other thinkers of the Early Modern Period the scientific rationality has been understood as follows. If any agent of rationality researches the whole, (s)he can find out its primary objects by the analytic method. These objects are separate and not mutually dependent. By compositions of these objects the whole can be explained. This intuition is embodied in the naïve set theory, where all sets are constructed by composition rules on the basis of atoms, mutually exclusive elements. So, any precise rigorous knowledge is considered a class of primary objects with relations among them.

Let A be a set of any nature. It is built up over atoms. Its powerset denoted by $\mathrm{P}(A)$ is defined as a family of all subsets of A. Let Ω be of the material universe consisting of things as atoms. Every member of $\mathrm{P}(\Omega)$ is called *event*. According to Descartes, the material universe is measurable. This means that each event E may have a characteristic number. Let this number $P(E)$ be called probability measure of E. Hence, $P(\cdot)$ is regarded as a *set function* (i.e., a function with sets constituting its domain).

The probability measure satisfies the following three axioms:

Axiom 1 (measurability): $0 \leq P(E) \leq 1$.

According to this axiom, the material universe is measurable.

Axiom 2 (certainty): $P(\Omega) = 1$.

This axiom says that there exists an event that takes place always and everywhere, i.e. there is an appropriate certain knowledge about the whole.

Axiom 3 (additivity): $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$

for any sequence of *mutually exclusive* events E_1, E_2, ... This axiom says that the probability of the *union* of *all* mutually exclusive events is the *sum* of their respective probabilities. In other words, for any set E there is its partition into mutually exclusive subsets E_1, E_2, ... such that their union gives E. For such subsets the probability measure is additive.

In statistical and econometric tools of business intelligence these axioms are basic, too. However, if we would like to involve quantitative methods to analyzing non-additive labels of symbolic interactions, we should avoid these axioms. In symbolic interactions we cannot define additive measures. Conventionally, probability measures run over real numbers of the unit [0, 1] and its domain is a Boolean algebra of $\mathrm{P}(\Omega)$ with atoms.

In order to define probability measures with a domain on events of the social world, we should appeal to the so-called *non-well founded sets* which do not have atoms at all. The main problem of these sets is that we cannot obtain a partition of sets in general case. Therefore we can preserve measurability without additivity. These new probability measures may be defined on non-Archimedean numbers, in particular on p-adic integers [15].

Let us recall that each p-adic number has a unique expansion $n = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k$, where $\alpha_k \in \{0,1,\ldots, p-1\}$, $\forall k \in \mathbf{Z}$, and $\alpha_{-N} \neq 0$, that is called the canonical expansion of p-adic number n. p-Adic numbers can be identified with sequences of digits:

$$n = \ldots \alpha_2 \alpha_1 \alpha_0, \alpha_{-1} \ldots \alpha_{-N}$$

The set of such numbers is denoted by $\mathbf{Q}_p$.

The expansion

$$n = \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=0}^{\infty} \alpha_k \cdot p^k,$$

where $\alpha_k \in \{0,1,\ldots, p-1\}$, $\forall k \in \mathbf{N}$, is called the *expansion of p-adic integer n*. This number sometimes has the following notation: $n = \ldots \alpha_3 \alpha_2 \alpha_1 \alpha_0$. The set of such numbers is denoted by $\mathbf{Z}_p$.

Extend the standard order structure on **N** to a partial order structure on p-adic integers (i.e. on $\mathbf{Z}_p$):

- for any p-adic integers σ, $\tau \in$ N we have $\sigma \leq \tau$ in N iff $\sigma \leq \tau$ in $\mathbf{Z}_p$,
- each finite p-adic integer $n = \ldots \alpha_3 \alpha_2 \alpha_1 \alpha_0$ (i.e. such that $\alpha_i = 0$ for any $i > j$) is less than any infinite number τ, i.e. $\sigma < \tau$ for any $\sigma \in$ N and $\tau \in \mathrm{Z}_p \setminus$ N.

Define this partial order structure on $\mathbf{Z}_p$ as follows:

O_{Z_p} Let $\sigma = \sigma_3 \sigma_2 \sigma_1 \sigma_0$ and $\tau = \tau_3 \tau_2 \tau_1 \tau_0$ be p-adic integers. (1) We set $\sigma < \tau$ if the following three conditions hold: (i) there exists n such that $\sigma_n < \tau_n$; (ii) $\sigma_k \leq \tau_k$ for all $k > n$; (iii) σ is a finite integer, i.e. there exists m such that for all $n > m$, $\sigma_n = 0$. (2) We set $\sigma = \tau$ if $\sigma_n = \tau_n$ for all $n = 0, 1, 2, \ldots$ (3) Suppose that σ, τ are infinite integers. We set $\sigma \leq \tau$ by induction: $\sigma \leq \tau$ iff $\sigma_n \leq \tau_n$ for all $n = 0, 1, 2, \ldots$ We set $\sigma < \tau$ if we have $\sigma \leq \tau$ and there exists n_0 such that $\sigma_{n0} < \tau_{n0}$.

This ordering relation is not linear, but partial, because there exist p-adic integers, which are incompatible. As an example, let p=2 and let σ represents the p-adic integer –1/3 = …10101…101 and τ the p-adic integer –2/3 = …01010…010. Then the p-adic streams σ and τ are incompatible. Now we can define sup and inf digit by digit. Then if $\sigma \leq \tau$, so $\inf(\sigma,\tau) = \sigma$ and $\sup(\sigma,\tau) = \tau$. The greatest p-adic integer according to our definition is –1 = …*xxxxxx*, where $x = p - 1$, and the smallest is 0 = …00000.

We can easily show that there is a set A of p-adic integers such that P(A) is not a Boolean algebra, therefore there is no partition of A (for more details see [15]):

Proposition 1. *Define union, intersection and complement in the standard way. The powerset* P(A), *where A is the set of p-adic integers, is not a Boolean algebra.*

Proof. Consider a counterexample on 7-adic integers. Let $A_1 = \{x : 0 \leq x \leq \ldots 11234321\}$ and $A_2 = \{x : \ldots 66532345 \leq x \leq \ldots 666666\}$ be subsets of $\mathbf{Z}_7$. It is readily seen that $\neg(A_1 \cap A_2) = \mathbf{Z}_7$, but $(\neg A_1 \cup \neg A_2) \subset \mathbf{Z}_7$, because

$$\neg A_1 = \{x : 11234321 < x \leq \ldots 666666\} \text{ and } \neg A_2 = \{x : 0 \leq x < \ldots 66532345\}.$$

Therefore $\mathbf{Z}_7 \setminus (\neg A_1 \cup \neg A_2) = A_3 = \{x : x = \ldots y_5 y_4 3 y_2 y_1 y_0$, where $y_i \in \{0, 1, \ldots, 6\}$ for each $i \in \mathbf{N} \setminus \{3\}\}$. It is obvious that the set A_3 is infinite. As a result, we obtain that $\neg(A_1 \cap A_2) \neq (\neg A_1 \cup \neg A_2)$ in the general case. Q.E.D.

Thus, indeed p-adic integers can be used for measuring non-additive labels of symbolic interactions, because on these numbers we cannot define additivity of probabilities in the conventional way.

Let us define p-adic probabilities as follows: a *finitely additive probability measure* is a set function $P_{\mathbf{Z}_p}(\cdot)$ defined for sets $E \subseteq \Omega$, it runs over the set $\mathbf{Z}_p$ and satisfies the following properties:

- $P_{\mathbf{Z}_p}(\Omega) = -1$ and $P_{\mathbf{Z}_p}(\varnothing) = 0$,
- if $A \subseteq \Omega$ and $B \subseteq \Omega$ are disjoint, i.e. $\inf(P_{\mathbf{Z}_p}(A), P_{\mathbf{Z}_p}(B)) = 0$, then $P_{\mathbf{Z}_p}(A \cup B) = P_{\mathbf{Z}_p}(A) + P_{\mathbf{Z}_p}(B)$.

Otherwise, $P_{\mathbf{Z}_p}(A \cup B) = P_{\mathbf{Z}_p}(A) + P_{\mathbf{Z}_p}(B) - \inf(P_{\mathbf{Z}_p}(A), P_{\mathbf{Z}_p}(B)) = \sup(P_{\mathbf{Z}_p}(A), P_{\mathbf{Z}_p}(B))$. Let us exemplify this property by 7-adic probabilities. Let $P_{\mathbf{Z}_p}(A) = \ldots 323241$ and $P_{\mathbf{Z}_p}(B) = \ldots 354322$ in 7-adic metrics. Then $P_{\mathbf{Z}_p}(A) + P_{\mathbf{Z}_p}(B) = \ldots 010563$; $\inf(P_{\mathbf{Z}_p}(A), P_{\mathbf{Z}_p}(B)) = \ldots 323221$; $(P_{\mathbf{Z}_p}(A) + P_{\mathbf{Z}_p}(B)) - \inf(P_{\mathbf{Z}_p}(A), P_{\mathbf{Z}_p}(B)) = \sup(P_{\mathbf{Z}_p}(A), P_{\mathbf{Z}_p}(B)) = \ldots 354342$.

- $P_{\mathbf{Z}_p}(\neg A) = -1 - P_{\mathbf{Z}_p}(A)$ for all $A \subseteq \Omega$, where $\neg A = \Omega \setminus A$.
- relative probability functions $P_{\mathbf{Z}_p}(A|B) \in \mathbf{Q}_p$ are characterized by the following constraint:

$$P_{\mathbf{Z}_p}(A|B) = -\frac{P_{\mathbf{Z}_p}(A \cap B)}{P_{\mathbf{Z}_p}(B)},$$

where $P_{\mathbf{Z}_p}(B) \neq 0$ and $P_{\mathbf{Z}_p}(A \cap B) = \inf(P_{\mathbf{Z}_p}(A), P_{\mathbf{Z}_p}(B))$.

The main originality of those probabilities is that conditions 2, 3 are independent. As a result, in a probability space $\langle {}^*\Omega, P_{\mathbf{Z}_p} \rangle$ some Bayes' formulas do not hold in the general case.

Thus, in defining p-adic probability measures the following axioms are used instead of Kolmogorov's axioms mentioned above [15]:

Axiom 1 (measurability): $0 \leq P_{\mathbf{Z}_p}(E) \leq -1$.

According to this axiom, the universe of social interactions is measurable.

Axiom 2 (certainty): $P_{Z_p}(\Omega) = -1.$

There is a certain knowledge about the whole. This axiom says that given enough information, the status of any event can be defined as certain. Otherwise, we face randomness, i.e. probability distributions representing our own lack of information.

Axiom 3 (non-additivity):

$$P_{Z_p}(\bigcup_{i=1}^{\infty} E_i) = \sup_{i=1}^{\infty} P_{Z_p}(E_i)$$

$$P_{Z_p}(\bigcap_{i=1}^{\infty} E_i) = \inf_{i=1}^{\infty} P_{Z_p}(E_i)$$

This means that I cannot divide the social universe into atoms. So, probability distributions cannot have additivity. Thus, data in symbolic interactions are randomized in non-Archimedean numbers and non-additively.

3. Basic Assumptions of Game Theory and Non-Additivity in Symbolic Interaction Games

The presuppositions of conventional probability theory, including the idea of additivity (partition into mutually disjoint subsets), are continued in game theory, where human behavior is understood as a step-by-step interaction of decision-makers for own payoffs. It is supposed that each player has a certain objective called payoff and takes actions deliberately in an attempt to achieve that objective. For this purpose each player takes into account knowledge or expectations of other decision-makers, because payoffs of different players can be in conflicts. So, the basic entity of game theory is a *player* who may be interpreted as an individual or as a group of individuals. A player is everyone who has an effect on others' payoffs. Notice that we can ever assume that a player participates in a symbolic interaction with others and then his/her behavior can be evaluated by non-Archimedean probability measures.

In game theory the sets of possible actions of individual players are considered primitives (atoms). Therefore we deal with a set Ω_G of all *strategies* (actions available to each player) in a game G such that Ω_G has a partition into mutually disjoint subsets E_1, E_2, ..., where each E_i contains actions of player i, $i = 1, 2, \ldots$ Each member $\langle e_1, e_2, \ldots\rangle$ of a set $E_1 \times E_2 \times \ldots$ is an outcome of the game and it is associated with *payoffs* $\langle a_1, a_2, \ldots\rangle$, where a_i is a payoff of player i after using a strategy e_i, $i = 1, 2, \ldots$ So, each player has own strategies and combinations with strategies of others give payoffs. Thus, the number of players is fixed and known to all parties. It is the first assumption of game theory, corresponding to the Descartes' hypothesis of additivity of labels in scientific investigations. We have mutually disjoint subsets E_1, E_2, ..., of Ω_G and each player knows such a partition. In other words, each player chooses among two or more possible strategies and knows how each strategy chosen by him/her or by another player determines the whole play.

Nevertheless, in symbolic interactions very often we face the situations when we do not know all players (e.g. lobbyists in politic games can be hidden), so we do not know an appropriate partition of Ω_G and it is possible that we do not know all strategies of Ω_G as such. In this case we can appeal to sets with a non-Archimedean ordering structure in the way of the previous section.

The second assumption of game theory, corresponding to the Descartes' hypothesis of additivity, is that all players are considered fully rational. It is understood as follows:

(i) Players know *all the rules of the game*, but it can appear that some rules change during the game. In symbolic interactions all depends on players and the symbolic meanings they produce within concrete interactions.

(ii) Players assume other parties to be fully rational. This means that all players have a *zero reflexion*: they know each other and know everything about each other including all strategies. Evidently, this assumption does not hold for symbolic interactions. I can cheat to hide my true motives and utter false announcements to lie. Therefore I cannot trust others.

(iii) All players attempt to *maximize their utility*. The latter means some ranking of the subjective welfare, when (s)he is ready to change something. As a result, all players *accept the highest payoffs*. Nevertheless, in symbolic interactions I can avoid the highest payoffs for the sake of some symbolic values, e.g. I can be altruistic or even sacrifice my life for somebody.

(iv) All players have *resistance points*, i.e. they can accept only solution's that are at or greater than their security levels. In symbolic interactions I can avoid this item, too, for the same reasons as in the previous item.

(v) All players *know the utilities and preferences of the other players* and develop tangible preferences among those options. *Preferences remain constant* throughout the game. But in symbolic interactions the players can lie and hide their true preferences or change their preferences through the interaction.

(vi) For any game there is *Pareto efficiency*. All players can take maximally efficient decisions which maximize each player's own interests. Let us recall that a distribution of utility A is called *Pareto superior* over another distribution B if from state B there is a possible redistribution of utility to A such that at least one player has the better payoff in A than in B and no player has the worse payoffs. In the situations of symbolic interactions when preferences may change through the game there is no Pareto efficiency in general case.

Due to all the assumptions of game theory mentioned above there are always common game solutions giving an endogenously stable or equilibrated state. These solutions are called *equilibria*. This term is extrapolated from physics, where it means a stable state in which all the causal forces internal to the system balance each other out unless it is perturbed by the intervention of some external force. So, game-theorists consider economic systems as mutually constraining causal relations, just like physical systems. These equilibria can be found out just by using the math tools of computations over payoffs. For symbolic interactions there are no equilibria in that meaning, but it is ever possible to reach a consensus that will be called a *performative equilibrium*.

If we use p-adic probability measures, we can appeal to other game-theoretic assumptions (for more details see [15]):

(i) Each *game* can be assumed *infinite*, because its rules can change.

(ii) Players can have *different levels of reflexion*: one player can know everything about another, but the second can know just false announcements from the first.

(iii) Some utilities can have *symbolic meanings*. These meanings are results of accepting symbolic values by some players. The higher symbolism of payoffs, the higher level of reflexion of appropriate players. On the zero level of reflexion, the payoffs do not have symbolic meanings at all. For consensus the players are looking for joint symbolic meanings.

(iv) *Resistance points* for players are reduced to the payoffs of the *zero level of reflexion*.

(v) The *joint symbolic meanings can change* through the game if a player increases his/her level of reflexion.

(vi) For any game there is *performative efficiency*, when all symbolic meanings of one player are shared by other players.

In case of these new game-theoretic assumptions we can calculate some aspects of symbolic interactions by probabilistic tools in non-Archimedean numbers [15].

4. Basic Assumptions for Statistical Tests and Econometric Models and Multimethodology

By means of *statistical tests* we can make inferences from samples to populations. These inferences are possible if the populations satisfy certain properties which are connected with the hypothesis of additivity. For example, in the *chi-square tests* it is assumed that the measure is taken on an interval or ratio scale and the population is considered normally distributed. In *F-tests* it is assumed that the variances of two populations are the same and estimations of the population variance are independent. So, statistical assumptions concern properties of statistical populations to allow us to draw conclusions on the basis of samples.

First of all, in any statistical tests there should be an *independence of observations from each other*. In other words, all the data obtained should be independent and randomly sampled. For instance, repeated measurements from the same people cannot be independent. The absence of correlation between data allows us to make partitions of data in accordance with additivity.

Statistical data should have a *normal distribution* (or at least be *symmetric*, when the graph of the data has the shape of a bell curve). The normal distribution is defined on scores in population in relation to two population parameters: (i) μ, the *population central tendency* (mean) (the *normality assumption*); (ii) σ, the *population standard deviation* (the *homogeneity* or *variance assumption*). Different normal distributions are obtained whenever the population mean or the population standard deviation are different. So, the normal distribution allows us to make standardized comparisons across different populations by their means and deviations. If the two means are the same, it is probably that, on the one hand, the populations are normally distributed and, on the other hand, we can check if the standard deviations (variances) are the same. If it is so, then the shared area under each of the population distribution curves will be constituted by all the area under the curves.

The *central limit theorem* of probability theory says that if the shared area occurs large enough we can suppose that the two populations are different in fact. Therefore no matter what distribution things have, the sampling distribution is normal if the sample is large enough, i.e. the estimate will have come from a normal distribution regardless of what the sample is. By contrast, if the shared area gets small we can suppose that the two populations become different.

At the end, in order to find out causal relations on the statistical data, we should create a linear correlation between the dependent and independent variables (this correlation is called *linear regression*). As a result, we can obtain a *model* that is *linear* in the parameters (i.e. in the coefficients on the independent variables):

$$y_i = b_1 x_{i1} + b_2 x_{i2} + \ldots + b_K x_{iK} + e_i \quad (i = 1, 2, \ldots, n),$$

where y_i is a dependent variable, x_i, x_{i2}, …, x_{iK} are independent variables, and b_1, b_2, …, b_K are parameters.

These models are used in *econometrics*. The basic properties of *classical linear regression model* are as follows:

(i) The variables cannot contain the same values for different observations (*sample variation*).

(ii) The observations should be randomly selected (*random sampling*), i.e. there should be no correlation between two different observations. This means that there is no autocorrelation in the error terms.

(iii) The mean of the error term e_i, given a specific value of the independent variable y_i, is zero (*zero conditional mean*).

(iv) The variance of the error term e_i is constant regardless of the regressors, i.e. the variance of the error term e_i does not depend on the value of independent variables (*no heteroscedasticity*).

(v) The error terms e_i and e_j of different serials are independently distributed so that their covariance is 0 (*no serial correlation* between the error term and exogenous regressors).

(vi) The error term e_i is normally distributed (*normally distributed errors*).

Let us notice that in actual experiments, we cannot generally obtain a perfect and consistent additive effect presented in a linear model. We are looking for observable regularity patterns to reduce them to statistical additivity. For example, in statistical tests the sample comes from an unknown population, therefore we do not know the standard deviation and thus we cannot calculate the standard error. But we do it. We try to generate causal evidence through econometric procedures like regression analysis, but this method is a reduction of real economic systems to physicalist models where all the causal forces considered internal to the system.

As we see, econometric models in business intelligence are based on the rigorous assumption of additivity that is a high abstraction put forward by the thinkers of the Early Modern Period. On the one hand, within this approach it is impossible to investigate all nuances of symbolic interactions in real human systems. On the other hand, there are no other approaches to find out causal relations in the real world. In order to solve this problem *multimethodology* (*mixed methods research*) has been proposed, where the collection and analysis of quantitative and qualitative data are combined. So, we carry out a quantitative research to assess magnitude and frequency of subject and we carry out a qualitative research to explore the meaning of subject. Among qualitative methods there are in-depth interviews, case study, introspections, focus groups, etc. for identification of previously unknown processes and explanations of why and how phenomena occur. Among quantitative methods there are tools for measuring pervasiveness of known phenomena and regularity patterns to make inferences of causality.

There are some design platforms for multimethodology (expert systems for mixed methods research), but these platforms are not automatic. Any combination of different methods is considered as complex of different steps fulfilled by different researches with different plans. Nevertheless, in non-Archimedean probabilities (section 2) and symbolic-interaction games (section 3) we can numerically calculate some sophisticated aspects of symbolic interactions such as reflexive games and performative efficiency. This means that we can build non-Archimedean extensions of models constructed by quantitative methods so that these extensions can express basic properties of symbolic interactions.

Conclusion

Thus, the basic assumptions of probabilistic, statistical and econometric tools in business intelligence are connected with the probability-theoretic hypothesis of additivity. In order to develop the mixed methods research combining qualitative and quantitative methods we can avoid this hypothesis and appeal to non-Archimedean probabilities (section 2) and symbolic-interaction games (section 3).

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References

1. Althusser, Louis and Balibar, Étienne. *Reading Capital*. London: Verso, 2009.
2. Arellano, M. *Panel Data Econometrics*. Oxford University Press, 2003.
3. Blumer, H.G. *George Herbert Mead and Human Conduct*. 2004.
4. Blumer, H.G. *Symbolic Interactionism: Perspective and Method*. 1969.
5. Bourdieu, P. *Acts of Resistance: Against the Tyranny of the Market*. New Press, 1999.
6. Bourdieu, P. and L.J.D. Wacquant. *An Invitation to Reflexive Sociology*. Chicago and London: Univ of Chicago Press, 1992.
7. Bourdieu, P. *Language and Symbolic Power.* Harvard University Press, 1991.
8. Bourdieu, P. *The Social Structures of the Economy*. Polity 2005.
9. *Collected Works of Karl Marx and Frederick Engels.* New York: International Publishers, 1988.
10. Davidson, R., J.G. MacKinnon. *Econometric Theory and Methods*. Oxford University Press, 2004.
11. McPherson, G. *Statistics in Scientific Investigation: Its Basis, Application and Interpretation*. Springer-Verlag, 1990.
12. Mead, G.H. *Mind, Self, and Society*. Ed. by Charles W. Morris. University of Chicago Press, 1934.
13. Mead, G.H. *Movements of Thought in the Nineteenth Century*. Ed. by C. W. Morris. University of Chicago Press, 1936.
14. Mead, G.H. *The Philosophy of the Act*. Ed. by C.W. Morris et al. University of Chicago Press, 1938.
15. Schumann, A. *Unconventional Logics in Decision Making*. Rzeszow: WSIiZ, 2014.
16. Wooldridge, J.M. *Econometric Analysis of Cross Section and Panel Data*. Cambridge MA, MIT Press, 2002.

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Co-constructive Logics for Proofs and Refutations

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Abstract:
This paper considers logics which are formally dual to intuitionistic logic in order to investigate a co-constructive logic for proofs and refutations. This is philosophically motivated by a set of problems regarding the nature of constructive truth, and its relation to falsity. It is well known both that intuitionism can not deal constructively with negative information, and that defining falsity by means of intuitionistic negation leads, under widely-held assumptions, to a justification of bivalence. For example, we do not want to equate falsity with the non-existence of a proof since this would render a statement such as "pi is transcendental" false prior to 1882. In addition, the intuitionist account of negation as shorthand for the derivation of absurdity is inadequate, particularly outside of purely mathematical contexts. To deal with these issues, I investigate the dual of intuitionistic logic, co-intuitionistic logic, as a logic of refutation, alongside intuitionistic logic of proofs. Direct proof and refutation are dual to each other, and are constructive, whilst there also exist syntactic, weak, negations within both logics. In this respect, the logic of refutation is weakly paraconsistent in the sense that it allows for statements for which, neither they, nor their negation, are refuted. I provide a proof theory for the co-constructive logic, a formal dualizing map between the logics, and a Kripke-style semantics. This is given an intuitive philosophical rendering in a re-interpretation of Kolmogorov's logic of problems.

Keywords: constructivism, paraconsistency, intuitionistic logic, co-intuitionistic logic, negation.

1. Constructive Truth and Falsity

1.1. Constructivism and undetermined statements

It is not altogether simple to pin down precisely what defines a logic as constructive. As is known, there are classically valid formulas that do not admit constructive proofs. More pressing, however, is the non-determinism that results from this lack of constructivity due to the appearance of multiple (disjunctive) succedents in typical presentations of classical logic (e.g Gentzen's LK). Constructive logics, by the disjunction property, ensure that a (cut-free) proof of $\alpha \vee \beta$ (where both α and β have single, cut-free proofs), must be a proof of one of the disjuncts[1]. Classically this breaks down since, by contraction, there may exist a disjunction $\alpha \vee \beta$ without any means of deciding which formula is proved. So, one may require that a constructive logic has the disjunction property, or some analogue of it.

This is a decent enough technical requirement, but it does not seem to get to the heart of constructivism. Rather, a key requirement on constructivism is typically that it admits a constructive account of truth, such for a proposition α to be "true" consists in there existing a constructive proof of α, and that, once true it remains so (monotonicity). However, pinning exactly what this requires is, notoriously tricky, resulting in a bifurcation of the understanding of what is meant by proof in the literature [17], [23], [27] into:

- Actualism (there actually exists a proof of α)
- Potentialism (α is potentially provable)[2].

Actualism is a tensed, time-bound, notion of proof given in a specific moment, and is typically thought to have been held by Brouwer and Heyting, see e.g. [13]. Potentialism is, rather, an objective notion of proof, typically held by Prawitz [21], and by Dummett (at certain points [9]). The open question here is whether or not for α to count as constructively true, actual possession of a proof of α is required of an agent, or merely an effectively decidable possible proof. The latter, (as pointed out in e.g. [16, pp.84–85]) seems to require commitment to an objective realm of propositions that ultimately faces similar objections to the classical account, leading to an "inert platonism of proofs" [6]. So, potentialism leads to a problematic account of truth regarding propositions that is both tenseless and independent of subjective understanding:

> [This] pays too much tribute to a "platonician" conception of mathematical inference: according to this conception, far from being deduced or extracted by ourselves, the consequences of an hypothesis follow from it by themselves, or rather in virtue of the existence of certain objects that it is none of our responsibility to conceive, or to make up, but only to discern; these objects may be likened to documents namely of documents already written, perhaps never read by anybody yet, but in which we could not fail to recognize, were we to become acquainted with them [...] By identifying proofs with sequences of formulas or, more generally, with objects that are independent from us, one almost unavoidably reduces the activity of justification to a scanning and control process that requires no cognitive or physical particular resource, and that consequently could not be affected by the limits of the cognitive capacities of the agents [5].

Potentialism, then, would not seem to adequately capture what is "constructive" regarding constructivism, which awards a key role to the actions of agents involved in the construction of a proof. But, the actualist position, depending upon how it is construed, may not alleviate these concerns. Insofar as the existence of a proof is understood in terms of a proof as "object", as, for example, Hilbert-style calculus seems to require, proofs will also be reduced to a form of ratification, where proofs are unaffected by our interaction with them.

To this end, we will work with a traditional, actualist, conception of proof which understands proofs in terms of "act" rather than object. It is clear, on Brouwer's account, for example, that *something occurs* whenever an agent comes to have an effective decision procedure for a proposition α: law of excluded middle becomes valid for α (e.g. [2]). Perhaps then, what is key to considering constructivism, is that formulas such as α may be reasoned with, considered, and so on, without the presumption of the existence of such a procedure, *and*, moreover, without even understanding α to be a proposition, at least insofar as a proposition is supposed to be semantically evaluable, and so on. If this is correct, then at the heart of constructivism, is that it allows for the existence of undecidability in the structure of syntax itself, and that our actions can render what is undecidable now to become decidable at some future moment. In this vein, we may follow Dubucs and Marion [6], who consider proof as "act", rather than "object":

> We propose that one distinguishes between two different notions of proof, namely those of proof as 'object' and as 'act'. According to the first conception, a proof is something like an assemblage of strings of symbols satisfying such and such property. From the second, more dynamic, conception, a proof is a process whose result may be represented or described by means of linguistic symbols.

This interpretation is appropriate to the typical BHK interpretation of logical connectives, which takes the justification of the assertion of a complex proposition to consist in the justification of its immediate constituents. This allows for a conception of proofs as acts, for which the BHK interpretation provides an account of their complex composition. In fact, this is clearer in Kolmogorov's [14] interpretation than even Brouwer's, which argues that constructive logic has to do "not with theoretical propositions but, on the contrary, problems".[3] In this setting, it is simple to see that what is central to constructivity is that it allows us to deal with undetermined formulas (or problems):

Undetermined: $(\exists\alpha)\alpha \not\vdash$ *and* $\neg\alpha \not\vdash$.

In terms of a standard Kripke structure, M as a triple $\mathsf{M} =_{df} (\mathsf{S}, \leq_+, \Vdash^+)$, where S is a collection of "states", or "stages", of reasoning, $\leq_+$ is a pre-order on S, and $\Vdash^+$ is a "forcing relation" between formulas of a language and elements of S. Let s_i indicate a stage of the construction, then, we have:

Undetermined: $(\exists\alpha), (\exists s_i \in \mathsf{S}), s_i \nVdash^+ \alpha$ *and* $s_i \nVdash^+ \neg\alpha$.

1.2. Asymmetry of truth and falsity

But, what is the status of undetermined formulas? There is an asymmetry between proof (and truth) and refutation (and falsity) in intuitionistic logic, which is particularly problematic from the point of view of constructive falsity. Undetermined formulas might simply be taken to be false, so that falsity is equated with non-truth. But, it does not seem right to say, of some formula α which is not provable at a stage s_1 but may become provable at a later stage s_2, that it is false, and then becomes true. Instead, falsity is typically equated with a reduction to a contradiction (this is the case in Heyting [13], and also Kolmogorov [14]), so that α is false whenever $\neg\alpha$ is true. This, too, is not unproblematic. As Shramko et al. [27] point out, if we attempt to formalize the definition of falsity in terms of negation, then we are led to a reliance, not only on a syntactic feature (negation), but also on truth, so there is a lack of independence of falsity. Moreover, as Dummett [7] is aware, this, together with potentialism is constructively suspect:

> There is a well-known difficulty about thinking of mathematical proofs [...] as existing independently of our hitting on them, which insisting that they are proofs we are capable of grasping or of giving fails to resolve. Namely, it is hard to see how the equation of the falsity of a statement (the truth of its negation) with the non-existence of a proof or verification can be resisted: but, then, it is equally hard to see how, on this conception of the existence of proofs, we can resist supposing that a proof of a given statement either exists or fails to exist. We shall then have driven ourselves into a realist position, with a justification of bivalence. If we refuse to identify falsity with the non-existence of a proof, we shall be little better off, because we shall find it

> hard to resist concluding that there are statements which are determinately neither true nor false [...] [6, p. 285].[4]

Kolmogorov provides us with a possible route around this issue by allowing that undetermined formulas are problems, not propositions. As such, these are just not the sort of thing that can be understood to be true or false. Rather, at the level of syntax, we have the capacity to deal with formulas that are undetermined, and, it is our action on syntax through which propositions become determined. This closely follows Martin-Löf's [16] argument that: 'A proof is, not an object, but an act [...], and the act is primarily the act as it is being performed, only secondarily, and irrevocably, does it become the act that has been performed.' In Kolmogorov's terms, this secondary object is the "solution" to a problem, and this is a proposition, and so truth-evaluable. An unsolved problem is not false (a problem is not the sort of thing that could be false), rather it is unproved, and so, non-accepted. Of course, this is to say that this state of a formula α must be non-monotonic in the sense that it must be possible to move from an unproved formula to a proved solution (but not back again). In other words, whilst the logic of problems is constructive, for solutions (propositions), law of excluded middle holds, so determination for a formula has a clearly link to truth-evaluability. We will return to this later.

Even if this is plausible, however, we are still left with two (connected) issues: one has to do with negation, and the other the asymmetry of truth. As is known, intuitionistic logic is incapable of refuting formulas directly. *Reductio ad absurdum* is capable only of conditionally proving negative statements, since $\neg\neg\alpha \rightarrow \alpha$ does not hold in general. So, even if we can show that $\neg\alpha$ leads us to a contradiction, we are not in a position to prove α. In addition, a negative answer to a problem will not, in general, return a solution. For example, if we answer "is α provable (at some stage s_i)?" negatively, this "no" represents something like a withdrawal that does not involve any commitment to a solution of α or $\neg\alpha$. But, this "no" can not be what is expressed by intuitionistic negation, since that is supposed to record commitment to $\neg\alpha$, whose truth is constructive (i.e. when we have a proof of $\neg\alpha$, this can not change over progressive stages of reasoning). This is problematic, since we do not, for example, want to say even of a mathematical statement that it is false if it is not proved, since then a statement such as "π is transcendental" would be deemed false prior to 1882. Then, to deal with statements which are provable, but not yet proved, requires a weaker than intuitionistic negation:

> Our reluctance to say that π was not transcendental before 1882, or, more significantly, to construe mathematical statements as significantly tensed, is not merely a lingering effect of platonistic misconceptions; it is, rather, because to speak in this way would be to admit into mathematical statements a non-intuitionistic form of negation, as will be apparent if one attempts to assign a truth-value to "pi is not algebraic" considered as a statement made in 1881. This is not because the "not" which occurs in "...is not true" or "...was not true" is non-constructive: we may reasonably view it as decidable whether or not a given statement has been proved at a given time. But, though constructive, this is an empirical type of negation, not the negation that occurs in statements of intuitionistic mathematics. The latter relates to the impossibility of ever carrying out a construction of some fixed type, the former to the outcome, at variable times, of some fixed observation or inquiry [8, p. 337].

Similarly, standard intuitionistic negation is too strong to express statements such that Goldbach's conjecture is undecided at present. What would be required, instead, is a weaker negation, which expresses that the statement lacks a proof at the present stage of reasoning. This weaker negation is precisely the "not" in the statements that Dummett analyses insofar as it records non-provability at a stage of reasoning. One of the properties of this weaker negation must be that, at some future stage of reasoning, the statement may be proved or refuted (and so, the problem solved). It seems, given the above arguments, that this is a key feature of any constructivism, and so, without it, we have an expressive inadequacy regarding the ability to express undeterminedness.

In this sense, typical construals of constructive negation appear as both too weak and too strong. On the one hand, it is not possible to directly establish that some formula α is refuted, rather, it is possible only to conditionally prove that, from assuming α, we can derive a contradiction. On the other, we seem to require a kind of "weak negation" where some potential proof of α does not go through, and so, expresses that no proof of α exists at that stage of reasoning, though without also precluding there being a proof of α or $\neg\alpha$ at some later stage. The former issue has to do with a symmetrical means by which it would be possible to directly refute some formula α; the second, with a means of recording a "withdrawal" or state of non-acceptability for α, which does not preclude later proof or refutation.

Consider the former issue in relation to Kolmogorov's interpretation. There, we have for some problem α, the asymmetric pair of questions: "is α provable?"; "is α reducible to a contradiction?". If, instead, we conceive of refutation as counterpart to proof, this would lead us to the symmetric pair: "is α provable?"; "is α refutable?". Whilst not strictly belonging to intuitionistic logic, this may be perfectly constructive. Indeed, Dummett [8] points out that it may be required outside of anything but a purely mathematical context:

> [A] proof of the negation of any arbitrary statement then consists of an effective method for transforming any proof of that statement into a proof of some false numerical equation. Such an explanation relies on the underlying presumption that, given a proof of a false numerical equation, we can construct a proof of any statement whatsoever. It is not obvious that, when we extend these conceptions to empirical statements, there exists any class of decidable atomic statements for which a similar presumption holds good; and it is therefore not obvious that we have, for the general case, any similar uniform way of explaining negation for arbitrary statements. It would therefore remain well within the spirit of a theory of meaning of this type that we should regard the meaning of each statement as being given by the simultaneous provision of a means for recognizing a verification of it and a means for recognizing a falsification of it, where the only general requirement is that these should be specified in such a way as to make it impossible for any statement to be both verified and falsified [7, p. 71].

Should we translate the stipulation into a BHK-style clause for negation, so that the negation of α is verified whenever α is falsified (and vice-versa), then this is analogous to Nelson's strong negation [19], and discussed in a similar context in [20].

In the following, we take a different tack, and consider the dual logic to intuitionistic logic, co-intuitionistic logic as a logic of refutation. This, I suggest, solves the first issue regarding "strong" negation, since this will be encoded in direct refutation. The second issue, regarding "weak" negation will be solved through consideration of the interaction between the two logics.

3. Constructive Co-intuitionistic Logic

In this section we provide an interpretation of co-intuitionistic logic as a constructive logic of refutation. To highlight the constructive nature of this logic, we note that the dual of "Undetermined" holds:

Inconsistent: $(\exists\alpha)\alpha \dashv \quad and \quad \neg\alpha \dashv$ (where $\dashv$ indicates a "refutation" relation).

Again, this may be understood constructively, as saying that there exists some formula α, for which neither α, nor its negation are refuted. This is not to say that $\alpha, \neg\alpha$ are considered "true", rather they too are undetermined in the sense that they are not yet refuted; they are "non-rejectable". Again, this can be expressed in terms of a Kripke structure for refutation, M as a triple $\mathsf{M} =_{df} (\mathsf{S}, \leq_{-}, \Vdash^{-})$, where S is a collection of "states", or "stages", of reasoning, $\leq_{-}$ is a pre-order on S, and $\Vdash^{-}$ is a refutation "forcing relation" between formulas of a language and elements of S. Let s_i indicate a stage of the construction, then, we have:

Inconsistent: $(\exists\alpha), (\exists s_i \in \mathsf{S}), s_i \nVdash^{-} \alpha \quad and \quad s_i \nVdash^{-} \neg\alpha$.

Here, there are problems which are undetermined, and so inconsistent (since neither are refuted). A refutation provides a way of determining a proposition, and we will show that the dual of the disjunction property, the conjunction property, ensures that law of non-contradiction holds for determined propositions. Understood this way, a paraconsistent logic of refutations admits constructivity just as an intuitionistic logic of proofs.

This is more perspicuous if we tackle this proof-theoretically.

Definition 1 (Languages S, S^d). *Over a denumerable set of atomic formulas, the languages for intuitionistic and co-intuitionistic logic are defined in Backus-Naur form* (α *is any atomic formula*):[5]

$$S\beta ::= [\alpha \mid (\neg_I \beta) \mid (\beta \wedge \beta) \mid (\beta \vee \beta) \mid (\beta \Rightarrow \beta)]$$

$$S^d\beta ::= [\alpha \mid (\neg_C \beta) \mid (\beta \wedge \beta) \mid (\beta \vee \beta) \mid (\beta \Leftarrow \beta)]$$

Here, $\neg_I$ and $\neg_C$ denote the negations of intuitionistic and co-intuitionistic negation, respectively, and $\Rightarrow$ and $\Leftarrow$ denote implication and co-implication, respectively.[6]

The usual construction of a deductive (or inference) structure involves their being a proof of some formula α, from a (possibly empty) set of formulas, Γ. Typically, we think of this as saying that α may be asserted in the context of Γ. So, intuitionism provides an account of the assertibility conditions for a set of formulas, with the general condition that a formula α is assertible whenever there is a proof of α. Here, we are also considering denial as the dual of assertion, and so we require an account of the deniability conditions for a set of formulas, perhaps with the general condition that a formula α is deniable whenever there is a refutation of α. To this end, we will consider a co-intuitionistic structure (as defined in [35]) as a refutation system, following the direction of [28], [29], [30]. The meaning of a sequent of the form $\Gamma \vdash_I \alpha$ is that, whenever there is a proof of Γ, there is also a proof of α (in intuitionistic logic). We also introduce ${}_C\dashv$, where the meaning of a sequent of the form $\Gamma \ {}_C\dashv \alpha$ is that, under the assumption that Γ is refuted, α is refuted also. Just as $\varnothing \vdash_I \alpha$ indicates that α is a theorem of an intuitionistic proof-theory, $\varnothing \ {}_C\dashv \alpha$ indicates that α is a "theorem" of a co-intuitionistic refutation-theory.

Remark 1. *Intuitionistically, implication internalizes the metalinguistic deducibility relation by means of the deduction theorem:* $\Gamma \vdash_I \alpha \quad \leftrightarrow \quad \vdash_L \bigcap \Gamma \Rightarrow \alpha$. *In co-intuitionistic logic, we can read co-implication as exactly dual:* $\Gamma \ {}_C\dashv \alpha \quad \leftrightarrow \quad {}_C\dashv \alpha \Leftarrow \bigcap \Gamma$.

With these in hand, we can define an inference structure L, and its dual, L^d, where L is understood to be a typical proof-structure, and L^d, a structure for refutation.

Definition 2 (Inference structure). *Define an inference structure* L *as an ordered pair,* $\langle S, \vdash_L \rangle$, *where* S *is as above, and* $\vdash_L$ *is a binary derivability relation between a subset of formulas of* S *(denoted* (Γ, Δ)*), and formulas of* S, *(denoted* (α, β)*), so* $\vdash_L \subseteq \mathcal{P}(S) \times S$.[7] *Say that* $\vdash_L$ *is normal when it is reflexive, transitive, monotonic and finitary. A sequent in* L *is an ordered pair* $\langle \Gamma, \alpha \rangle$ *where* Γ *is a set of formulas and* α *a single formula (and* $\Gamma \cup \{\alpha\}$*).*

The dual notion of refutation is now simple to define.

Definition 3 (Sequent duality). *The dual of* $\Gamma \vdash_L \alpha$ *is* $\Gamma^d \ {}_{L^d}\dashv \alpha^d$ *(for any* $\Gamma \subseteq S, \alpha \in S$ *and* $\Gamma^d \subseteq S^d, \alpha^d \in S^d$*).*

Definition 4 (Dual inference structure). *Define a dual inference structure* L^d *as an ordered pair,* $\langle S^d, {}_{L^d}\dashv \rangle$, *where* S^d *is as above, and* ${}_{L^d}\dashv$ *is a binary derivability relation between a subset of formulas of* S^d *and formulas of* S^d, *so* ${}_{L^d}\dashv \subseteq \mathcal{P}(S^d) \times S^d$. *Again, say that* ${}_{L^d}\dashv$ *is normal when it is reflexive, transitive, monotonic and finitary. A sequent in* L^d *is an ordered pair* $\langle \Gamma, \alpha \rangle$ *where* Γ *is a set of formulas and* α *a single formula (and* $\Gamma \cup \{\alpha\}$*), bearing in mind that these belong to* S^d.

In foregrounding the notion of refutation, we are interested in the conditions under which α is refutable given a set of (denied) assumptions Γ, thus allowing for a non-trivial account of the deniability of α in the context of Γ. Of course, if we were working in classical logic, then L and L^d may be dealt with utilizing a multiple-conclusion structure such as Gentzen's LK, which is self-dual.[8] But, here, we are interested in the non-self-dual, but, nonetheless, dual structures, intuitionistic logic and co-intuitionistic logic. We briefly sketch the details, closely following the presentation found in [35]. I assume familiarity with the system LJ, denoted I, which is obtained through the restriction of symmetric sequents to sequents in which a multiset of formulas may appear in the antecedent, but (at most) a single formula in the consequent. Analogously, the system LDJ is usually obtained through a restriction so that (at most) a single formula may appear in the antecedent. However, in distinction to the typical presentation in, e.g. [35], due to the way in which we are reading refutation sequents $\Gamma^d \;{}_C\!\dashv \alpha^d$, this will amount to a restriction to singularity on the right.

Definition 5 (The system LDJ, denoted C). *Since all formulas are constructed from S^d, I will drop the superscript in what follows, apart from the identity axiom.*

$$\alpha^d \;{}_C\!\dashv \alpha^d \quad \text{(Id)}$$

$$\frac{\Delta \;{}_C\!\dashv}{\Delta \;{}_C\!\dashv \alpha}\ (\text{Thin-}R) \qquad\qquad \frac{\Delta \;{}_C\!\dashv \beta}{\Delta, \alpha \;{}_C\!\dashv \beta}\ (\text{Thin-}L)$$

$$\frac{\Delta, \alpha, \alpha \;{}_C\!\dashv \beta}{\Delta, \alpha \;{}_C\!\dashv \beta}\ (\text{Cont})$$

$$\frac{\Delta, \alpha, \sigma, \Theta \;{}_C\!\dashv \beta}{\Delta, \sigma, \alpha, \Theta \;{}_C\!\dashv \beta}\ (\text{Int}) \qquad\qquad \frac{\Delta, \alpha \;{}_C\!\dashv \beta \quad \Gamma \;{}_C\!\dashv \beta}{\Delta, \Gamma \;{}_C\!\dashv \beta}\ (\text{Cut})$$

$$\frac{\Delta \;{}_C\!\dashv \alpha}{\Delta \;{}_C\!\dashv \alpha \wedge \beta}\ (\wedge R_1) \qquad\qquad \frac{\Delta \;{}_C\!\dashv \beta}{\Delta \;{}_C\!\dashv \alpha \wedge \beta}\ (\wedge R_2)$$

$$\frac{\Delta, \alpha \;{}_C\!\dashv \sigma \quad \Delta, \beta \;{}_C\!\dashv \sigma}{\Delta, \alpha \wedge \beta \;{}_C\!\dashv \sigma}\ (\wedge L)$$

$$\frac{\Delta \;{}_C\!\dashv \alpha \quad \Delta \;{}_C\!\dashv \beta}{\Delta \;{}_C\!\dashv \alpha \vee \beta}\ (\vee R)$$

$$\frac{\Delta, \alpha \;{}_C\!\dashv \sigma}{\Delta, \alpha \vee \beta \;{}_C\!\dashv \sigma}\ (\vee L_1) \qquad\qquad \frac{\Delta, \beta \;{}_C\!\dashv \sigma}{\Delta, \alpha \vee \beta \;{}_C\!\dashv \sigma}\ (\vee L_2)$$

$$\frac{\Delta, \beta \;{}_C\!\dashv \alpha}{\Delta \;{}_C\!\dashv \alpha \Leftarrow \beta}\ (\Leftarrow R) \qquad\qquad \frac{\Delta, \alpha \;{}_C\!\dashv \sigma \quad \Gamma \;{}_C\!\dashv \beta}{\Delta, \Gamma, \alpha \Leftarrow \beta \;{}_C\!\dashv \sigma}\ (\Leftarrow L)$$

$$\frac{\Delta, \alpha \;{}_C\!\dashv}{\Delta \;{}_C\!\dashv \neg_C \alpha}\ (\neg_C R) \qquad\qquad \frac{\Delta \;{}_C\!\dashv \alpha}{\Delta, \neg_C \alpha \;{}_C\!\dashv}\ (\neg_C L)$$

Theorem 1. *The rule Cut is eliminable from the system C.*

Proof. The proof is routine and can be found in [35]. □

As is known, Cut is eliminable from the system I .

Corollary 1. *The system* C *has the subformula property and is consistent.*

The constructivity of co-Intuitionistic logic (understood as a logic of refutation) is closely related to this conjunction property:

$$_{C}\dashv \alpha \wedge \beta \quad \textit{iff} \quad _{C}\dashv \alpha \quad \textit{or} \quad _{C}\dashv \beta .$$

Theorem 2. *Co-Intuitionistic logic* (*as refutation*) *has the conjunction property.*

Proof. The proof follows the typical one given for the disjunction property in I . Say that there exists a cut-free proof of $_{C}\dashv \alpha \wedge \beta$, then, since it is not an axiom, the proof must end with a rule. This must be one of either $\wedge R_1$ or $\wedge R_2$, and so from either $_{C}\dashv \alpha$ or $_{C}\dashv \beta$. The other direction is obvious. □

We can now precisely define the duality between the intuitionistic and co-intuitionistic inference structures.

Definition 6. *We can define a mapping from* S *to* S^d (*and back*) $(-)^d : S \leftrightarrow S^d$:

$$\alpha^d : \alpha \textit{ for atomic} \alpha$$

$$(\alpha \wedge \beta)^d : \alpha^d \vee \beta^d$$

$$(\alpha \vee \beta)^d : \alpha^d \wedge \beta^d$$

$$(\neg_I \alpha)^d : \neg_C (\alpha)^d$$

$$(\alpha \Rightarrow \beta)^d : \beta^d \Leftarrow \alpha^d$$

$$\{\alpha_1 ... \alpha_n\}^d : \{\alpha_1^d ... \alpha_n^d\}$$

$$(\varnothing \vdash_L \alpha)^d : \varnothing \;_{L^d}\dashv \alpha^d$$

$$(\Gamma \vdash_L \alpha)^d : \Gamma^d \;_{L^d}\dashv \alpha^d$$

Here, $(-)^d$ is a mapping that is involutive, and without fixed points, operating as a means of expressing the duality relationship between proofs and refutations.[9]

The above can be interpreted in a "falsity-preserving" Kripke-structure as follows.

Definition 7 (Co-intuitionistic Kripke-structure). *We define a refutation Kripke structure,* M *as a triple* $\mathsf{M} =_{df} (\mathsf{S}, \leq_{-}, \Vdash^{-})$*, where* S *is a collection of "states", or "stages", of reasoning,* $\leq_{-}$ *is a pre-order on* S*, and* $\Vdash^{-}$ *is a refutation "forcing relation" between formulas of a language and elements of* S*. Let* s_i *indicate a stage of the construction where* $i, 1 \leq_{-} i \leq n$*. Then, at any stage,* s_i*, we define:* $V^{-}(\alpha) = \{\alpha \in S : s_i{}^{-}\alpha\}$.

For compound formulas, the following conditions hold:

(1) $[\wedge] s_i \Vdash^{-} (\alpha \wedge \beta)$ *iff* $\alpha \in (V^{-}, s_i)$ *or* $\beta \in (V^{-}, s_i)$

(2) $[\vee] s_i \Vdash^{-} (\alpha \vee \beta)$ *iff* $\alpha \in (V^{-}, s_i)$ *and* $\beta \in (V^{-}, s_i)$

(3) $[\neg_C] s_i \Vdash^{-} (\neg_C \alpha)$ *iff* $\forall s_i' \, and \, s_i \leq_{-} s_i', \alpha \notin (V^{-}, s_i')$

(4) $[\Leftarrow] s_i \Vdash^{-} (\alpha \Leftarrow \beta)$ *iff* $\exists s_i' \, and \, s_i \leq_{-} s_i', \alpha \in (V^{-}, s_i') \, or \, \beta \notin (V^{-}, s_i')$.

Remark 2. *Note that this interpretation is atypical in relation to other interpretations of co-Intuitionism since we take are considering it from the point of view of constructivity over* V^{-}*. So, the above clauses tell us that, for example, a refutation of* $(\alpha \wedge \beta)$ *consists of a refutation of* α*, or a refutation of* β*. Most interestingly, perhaps, is that, in typical constructions of co-intuitionistic logics (and many other paraconsistent logics), there is no operator definable that is an analogue to modus ponens. As is clear, this is not the case in our refutation interpretation.*

Lemma 1. *The co-intuitionistic logic is sound and complete w.r.t the Kripke structure for refutation.*

Proof. This is easily checked by simply dualizing the standard approach to the Kripke structure for intuitionistic logic. □

Algebraically, a semantics can easily be given in a Brouwerian (sometimes called co-Heyting) algebra, which is the dual to Heyting algebra, e.g. [11]. Moreover, [17, §11] shows that co-intuitionistic logic is the internal logic of a complement topos, which is dual to a topos (whose internal logic is intuitionist). The former is a category, with terminal object 1, defined as an object Ω with an arrow $F : 1 \to \Omega$, and where F is dual to T in an ordinary topos.[10] This suggests that co-intuitionistic logic, interpreted as a constructive logic of refutations, may well be valuable in its own right.

2.1. Negation, again

Let us consider the interpretations of negation for intuitionistic and co-intuitionistic logic, respectively, in order to determine the relation between the two:

(1) $[\neg_I]s_i \|\!-^{+} (\neg_I \alpha) \quad iff \quad \forall s_i' \; and \, s_i \leq_{-} s_i', \alpha \notin (V^{+}, s_i')$

(2) $[\neg_C]s_i \|\!-^{-} (\neg_C \alpha) \quad iff \quad \forall s_i' \; and \, s_i \leq_{-} s_i', \alpha \notin (V^{-}, s_i')$

Both of the above correspond to the idea that negation can be given an intuitive reading in terms of the respective implications of the two structures. Just as intuitionistic negation is typically defined as $\neg_I \alpha : \alpha \rightarrow \perp$, where $\perp$ is a contradictory statement, so co-intuitionistic negation is typically defined as $\neg_C \alpha : \mathrm{T} \leftarrow \alpha$, where T is a tautologous statement. This reading of the respective negations ensures that $(\neg_I \alpha)$ can never be proved, and $\neg_C \alpha$, can never be refuted. So, one, perhaps obvious, suggestion would be to tie the two logics together by means of negation, so that we define the two negations along the lines suggested by Dummett:

(3) $[\neg_I]s_i \|\!-^{+} (\neg_I \alpha) \quad iff \quad \forall s_i' \; and \, s_i \leq_{-} s_i', \alpha \in (V^{-}, s_i')$

(4) $[\neg_C]s_i \|\!-^{-} (\neg_C \alpha) \quad iff \quad \forall s_i' \; and \, s_i \leq_{-} s_i', \alpha \in (V^{+}, s_i')$

That is, the negations of the two structures simply tells us that the negation of a proof intuitionistically, is equivalent to a refutation, co-intuitionistically.[11] But, simply adding constructive refutation to intuitionistic logic, however, would do nothing to alleviate the issue discussed above regarding "weak" negation. In general, if we accept the metalinguistic restriction that no statement can be both proved and refuted, no syntactic negation need even be involved in that relationship. Moreover, far from disappearing, the problem of asymmetry will now rear its head again, since the idea of "non-acceptability" for a formula, which corresponds to "not yet determined", has no counterpart for "non-rejectability" given that refutation is now a direct notion. What seems to be required, is an interpretation of a syntactic negation operator which, attached to a formula α expresses the idea that a proof of α "does not go through", and so, leaves α undetermined.[12] This appears perfectly possible, now that we have expanded our original language with the dual language for co-intuitionistic logic.

Further still, as mentioned above, the typical intuitionistic negation defined by introducing a constant, $\perp$, whilst fairly simple to interpret mathematically (as an arithmetic absurdity, say), is far trickier in general. In fact, as pointed out in [3], even in the domain of arithmetic, there appears to be an ineliminable (and vicious) circularity in the typical definition of negation as $\alpha \rightarrow 0 = 1$.[13] Similarly, Tennant [32] argues against the view that proofs of negation can be understood as a proof of a false sentence (such as $\perp$). Instead, Tennant takes a view similar to Brouwer's, that negation can be understood in terms of a "construction that no longer goes", suggesting that negation indicates a "dead-end" of a construction. The difficulty with these attempts to construe the "correct" definition of constructive negation lies with the fact that the desiderata pulls us in different directions. But, since we are now allowing for direct refutation to exist within the co-intuitionistic structure, we might consider, in addition, a pair of "weak negations", which are defined purely by means of proof-theoretic constraints upon contrariety and sub-contrariety, respectively.[14]

These “weak” negation can then be interpreted as:

(5) $\quad [\neg_I]s_i \Vdash^{+} (\neg_I \alpha) \quad \textit{iff} \quad \alpha \notin (V^{+}, s_i)$

(6) $\quad [\neg_C]s_i \Vdash^{-} (\neg_C \alpha) \quad \textit{iff} \quad \alpha \notin (V^{-}, s_i^{'})$

Of course, the standard negation as derivation of a contradiction (or tautology) does this, but this “reaches” across stages to index the impossibility of α being forced at any stage. The weaker negation allows that α remains an open problem, and so can make sense of the status of claims such as “π is transcendental” prior to 1882. Of course, the status of weakly negated formulas can not then be monotonic, but, as [29, §2.4] point out, adding such an operator to standard intuitionistic logic would be conservative since its existence has no impact on the usual interpretation of all other connectives.[15] The same is true for co-intuitionistic logic (as structure of refutation), as can easily be checked.

Now, since we already have direct refutation in the co-intuitionistic logic of refutation, we have that the refutation of α is a solution (and this is monotonic in the sense that, when α is refuted, it remains so), whilst $\neg_I \alpha$ corresponds to the assumption of α leading to a non-proof, and so returns an “unsolved” problem (and this is non-monotonic); and $\neg_C \alpha$ corresponds to the assumption of α leading to a “non-refutation”, so similarly returning an “unsolved” problem. To clarify this, let us now consider the relation between the two logics for proofs and refutations.

3. Interpreting Problems and Solutions

On the above, we have a co-constructive logic for proofs and refutations, the individual structures of which contain “weak” negations capable of expressing “non-proved” and “non-refuted”, respectively, in addition to a mapping $(-)^d$. The latter is an involution which is permissible at any time, and corresponds to a switching of approach between “prover” and “refuter”. In other words, on the level of problems, we are permitted, at any stage by means of $(-)^d$ to change our reference frame from approaching a formula from the point of view of attempting to prove it, or refute it. Moreover, whilst this mapping is involutive, it does not collapse to classical negation for two reasons. First, we understand $(-)^d$ to be an involution without fixed points, and second, we have syntactically separated the language. The latter means that certain “formulas” are barred in the sense that they can not be “well formed”. For example, $\neg_I \alpha^d$ and $\neg_C \alpha$ are not *wff* 's, and so neither is $\neg_I \neg_C \alpha$. So, the two structures, whilst allowed to communicate via $(-)^d$, are also kept separate. Otherwise, as is known, the combined structure will collapse to a single partial order.[16]

As should be obvious, then, the mapping $(-)^d$, whilst it obviously has many of the properties of the “strong negation” discussed above, can not be identified with it. Rather, it will be the case that, only under certain circumstances, $(-)^d$ plays the role of strong negation. In fact, as we shall see, the circumstances in which this is possible corresponds exactly to those in which are dealing with propositions, or solutions, rather than problems. In this sense, strong negation operates metalinguistically as a way of demarcating a fundamental relation of preclusivity between a direct proof and a direct refutation. Let us consider this is more detail.

The weak negations may be interpreted in standard Kripke-interpretations as: $s_1 \Vdash^{+} \neg_I \alpha$ when "α is not amongst the set of proved statements at stage s_1"; and, $s_1 \Vdash^{-} \neg_C \alpha$ as "α is not amongst the set of refuted statements at stage s_1". There are, therefore, four "modes" of proof and refutation that are characterizable in the structure as a whole:

(3.1) $\vdash_I \alpha \leftrightarrow$ Direct proof of α

(3.2) ${}_C\dashv \alpha \leftrightarrow$ Direct refutation of α^d

(3.3) $\alpha \vdash_I \;\leftrightarrow$ Indirect proof that α is unproved

(3.4) $\alpha \;{}_C\dashv \;\leftrightarrow$ Indirect refutation that α^d is unrefuted

The relationship between direct and indirect potential proofs (refutations) can be represented by the following diagram, (α_{up} indicates that α is not proved; α^d_{ur} that α^d is unrefuted):

$$\begin{tikzcd} \alpha \arrow[r, "\neg_I"] \arrow[dr, leftrightarrow] & \alpha_{up} \arrow[dl, leftrightarrow] \\ \alpha^d_{ur} \arrow[ur, phantom, "d"] & \alpha^d \arrow[l, "\neg_C"] \end{tikzcd}$$

Remark 3. *The resultant structures for I and C can now be better understood in relation to the properties of paracompleteness and paraconsistency, respectively. Due to the way in which we have constructed the two, it is not the case that law of non-contradiction is refutable in I, nor is law of excluded middle provable in C. Rather, there exist only indirect proofs (refutations) that they are unproved (unrefuted). Furthermore, the non-derivability of law of excluded middle in I is exactly mirrored by the non-derivability of law of non-contradiction in C:*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \vdash_I \alpha}(Axiom)}{\vdash_I \alpha, \neg_I \alpha}(\neg_I R)}{\vdash_I \alpha, \alpha \vee \neg_I \alpha}(\vee R)}{\vdash_I \alpha \vee \neg_I \alpha, \alpha \vee \neg_I \alpha}(\vee R) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \;{}_C\dashv \alpha}(Axiom)}{{}_C\dashv \alpha, \neg_C \alpha}(\neg_C R)}{{}_C\dashv \alpha, \alpha \wedge \neg_C \alpha}(\wedge R)}{{}_C\dashv \alpha \wedge \neg_C \alpha, \alpha \wedge \neg_C \alpha}(\wedge R)$$

In both cases, non-derivability results from the prohibition of contraction on the right of $\vdash_I$ and ${}_C\dashv$. Moreover, just as $\neg_I(\alpha \vee \neg_I \alpha) \vdash_I$ is derivable intuitionistically, so too is $\neg_C(\alpha \wedge \neg_C \alpha) \;{}_C\dashv$ co-intuitionistically.

Now, consider again the distinction between problems and solutions. It is natural, in this setting to take a conclusive proof (refutation) to mark the end-point of a questioning process: "is α provable?", "is α refutable?". A "solution", to borrow Kolmogorov's terminology, is a determinate and conclusive answer to these questions. Of course, a problem can not be true nor false, and neither can it be asserted nor denied. For a proposition, α, however, we should be able to say of α

that it "holds", that α is true (false), and so on. We can think of a proposition $\overline{\alpha}$ as the result, or solution, to a problem, which terminates attempts to answer that problem. Then, the mapping $\overline{(-)}$: $\alpha \to \overline{\alpha}$ and $\alpha^d \to \overline{\alpha^d}$ will denote that we have moved from consideration of α as a problem to a conclusive solution of that problem.

We note two features of conclusive proofs (refutations). First, conclusive proofs (refutations) will be constructive since it is not possible to destroy a conclusive proof (refutation) at any later stage of reasoning. Second, the relation between conclusive proofs and conclusive refutations is one of preclusivity, since it is not possible for their to exist a refutation of α whenever there exists a solution which says that α is proved. It is, therefore, at this point that we can allow I and C to directly communicate, such that for some α, we have that either $\varnothing \vdash \alpha$ or $\varnothing \dashv \alpha^d$, and $(\varnothing \vdash \alpha) \cap (\varnothing \dashv \alpha^d) = \varnothing$.

To clarify, let us briefly define a partial Kripke-structure for conclusive proofs and refutations.

Definition 8. *Define a (partial) Kripke-structure for conclusive proofs and refutations as the combined structure* $\mathsf{M} =_{df} (\mathsf{S}, \leq_+, \leq_-, V^+, V^-)$. *We let* $\models^+$ *and* $\models^-$ *denote constructive forcing relations over conclusive proof and refutation, respectively. Then, at any stage,* s_i, *we define:* $V^+(\alpha) = \{\alpha \in S : s_i{}^+\alpha\}$ *and* $V^-(\alpha) = \{\alpha \in S : s_i{}^-\alpha\}$. *Furthermore,* $\models^+$ *and* $\models^-$ *are constructive for both* V^+ *and* V^-, *satisfying the following persistence properties:*

If $s_1 \leq_+ s_2$ then $\forall \alpha \in (V^+, s_1), \alpha \in (V^+, s_2)$; and If $s_1 \leq_- s_2$ then $\forall \alpha \in (V^-, s_1), \alpha \in (V^-, s_2)$.

This ensures that, whenever a formula is conclusive assertible (deniable) at a stage, that "state" remains at every stage upstream, so, if $s_1 \models^+ \alpha$, then $s_2 \models^+ \alpha$; if $s_1 \models^- \alpha$, then $s_2 \models^- \alpha$.

Theorem 3. $\leq_+$ *and* $\leq_-$ *are partial orders, which are reflexive, monotonic and transitive.*

Proof. Obvious. □

The clauses defining compound formulas are relatively standard (see e.g. [20]), but unlike standard Kripke-structures, there is no clause for negation since that can only ever be involved in an indirect proof (refutation).[17]

Definition 9 (Interpretation for compound formulas for conclusive proofs and refutations):

(7) $[\wedge] s_i \models^+ (\alpha \wedge \beta)$ *iff* $\alpha \in (V^+, s_i)$ *and* $\beta \in (V^+, s_i)$

(8) $[\wedge] s_i \models^- (\alpha \wedge \beta)$ *iff* $\alpha \in (V^-, s_i)$ *or* $\beta \in (V^-, s_i)$

(9) $[\vee] s_i \models^+ (\alpha \vee \beta)$ *iff* $\alpha \in (V^+, s_i)$ *or* $\beta \in (V^+, s_i)$

(10) $[\vee] s_i \models^- (\alpha \vee \beta)$ *iff* $\alpha \in (V^-, s_i)$ *and* $\beta \in (V^-, s_i)$

(11) $[\Rightarrow] s_i \models^{+} (\alpha \Rightarrow \beta)$ *iff* $\forall s_i'$ *and* $s_i \leq s_i', \alpha \notin (V^{+}, s_i')$ *or* $\beta \in (V^{+}, s_i')$

(12) $[\Leftarrow] s_i \models^{-} (\alpha \Leftarrow \beta)$ *iff* $\forall s_i'$ *and* $s_i \leq s_i', \alpha \in (V^{-}, s_i')$ *or* $\beta \notin (V^{-}, s_i')$

We also note the following restriction on these structures, given preclusivity:

Definition 10 (Restrictions on Kripke-structures)

$$\forall s_i \in S, \alpha \in (V^{+}, s_i) \leftrightarrow \quad \alpha \notin (V^{-}, s_i)$$
$$\forall s_i \in S, \alpha \in (V^{-}, s_i) \leftrightarrow \quad \alpha \notin (V^{+}, s_i)$$

Moreover, we have a metalinguistic device of a "strong" negation at the level of solutions. So, for example, we can say that, for any semantic interpretation of solutions, and any "solved" proposition α, that either $\alpha \in V^{+}$, or $\alpha \in V^{-}$, and, for no α is $\alpha \in V^{+}$ and $\alpha \in V^{-}$. If we think of this in terms of the development of constructive theories over time, then we have the relatively simple procedure of just "adding" a solution to the appropriate set of sentences that are either proved or refuted by that stage in the development of the theory. The discrepancy between typical construals of negation, then, is that "strong" negation is applicable only to solutions, whilst there also exist two, dual, weak negation operators that are applicable only to problems. Such a demarcation may not be to everybody's taste, but it allows us to capture what is constructive about constructive logic.

It is simple to give an algebraic semantics for solutions in terms of the characteristic functions of the set of proved and refuted sentences.

Definition 11 (Partial semantic structure). *Define a partial semantic structure as an ordered pair* $\langle S, V \rangle$, *where* S *is an enumerable set of formulas, and* V *is a valuation space representing a structure of admissible valuations on the language.*[18] *Then, let,* $\mathsf{V} = \{1,0\}$ *be a set of truth-values, and* $\mathsf{D} = \{1\}$ *be a proper subset of* V *which is a designated value. A valuation* v *is a partial function on* S *assigning a truth-value* $\in \mathsf{V}$ *to a formula* $\alpha \in S$, *where* $v : S \rightarrow \{\mathsf{V}\}$.

The valuation function is extended by induction on the complexity of the Kripke clauses given above. Then, if we let Γ' denote the set of propositions which are proved at a stage s, and Δ' the set of propositions that are refuted at s:

Definition 12 (Characteristic function). *For any* $\langle \Gamma', \Delta' \rangle$ *we take a valuation to be the characteristic function such that:*

$$v(\Gamma') = \{v : v(\alpha) = 1 \textit{ for each } \alpha \in \Gamma'\}; \text{ and } v(\Delta') = \{v : v(\alpha_i) = 0 \textit{ for each } \alpha_i \in \Delta'\}.$$

Lemma 2. *We define a valuation* v *over formulas* $\alpha \in S$ *as* $\{v : \alpha \rightarrow \{1,0\}\}$, $v(\alpha) = 1 \leftrightarrow \varnothing \vdash_I \alpha$, $v(\alpha) = 0 \leftrightarrow \varnothing \; {}_C\!\dashv \alpha$. *Then,* $\langle S, V \rangle$ *is simply the collection of all such functions.*

The resultant semantic structure is obviously sound and complete with respect to the *solutions* of the combined structure. In fact, since we can think of each stage of the construction as constituting a partial interpretation where the formulas of Γ_i are mapped to the valuation $\{1\}$ in a semantic structure, and the formulas in Δ_i to $\{0\}$, we have that, for every atomic α, which is a solution;

Either $\exists\Gamma'(\Gamma' \mid\!-_I \alpha)$ or, $\exists\Delta'(\Delta' \;_C\!-\!\mid \alpha)$.

3.1. "Not-yet" proved or refuted

What should we say, then, regarding problems that are not yet solved? There, we have problems which are not-yet actualized as solutions. This brings with it an obvious relation with the above distinction between actualism and potentialism. In effect, it is possible to understand problems as a space of potential proofs (refutations), and solutions as actual proofs (refutations). Moreover, we might also view the syntactical operations of co-constructive logics as properly lying "beneath" what is, in essence, a bivalent determination of propositions. Since, if, for example, we allow propositions to be identified with their proofs (refutations) in a simple extension of Brouwer's position, then bivalence holds for all propositions as is simple to see from the above semantic interpretation of solutions. This carries over from a translation of Dummett's argument that no statement may be both verified and falsified, together with the fact that a decision procedure holds for every solution. We can put this as a principle as follows:

Coherence: *For no proposition* $\bar{\alpha}$ *is it the case that* $\alpha \in V^+$*, and* $\alpha \in V^-$*; and for each* $\bar{\alpha}$*, either* $\alpha \in V^+$ *or* $\alpha \in V^-$.

This much is standard, and, if it were to hold unrestrictedly for all formulas, brings with it a form of classical logic. What is important, however, from the point of view of constructivism, is that coherence does not hold across the board. Rather, our co-constructive logic allows us to deal, in addition, with states that are "paracoherent"[19]:

Paracoherence: *Coherence holds, restrictedly, for solutions, whilst otherwise the structure of proofs and refutations* (*for problems*) *may be non-trivially undetermined and inconsistent*.

In fact, I would go so far as to suggest that this captures the raison d'ętre of constructive logic.

References

1. Bellin, G., Carrara, M., Chiffi, D., and Menti, A. Pragmatic and dialogic interpretations of bi-intuitionism. Logic and Logical Philosophy, 2014.
2. Brouwer, L. *L.E.J. Brouwer: collected works*. Amsterdam: North-Holland Publishing Company, 1975.
3. Cook, R. T. and Cogburn, J. What negation is not: Intuitionism and '0=1', *Analysis*, 60(265):5–12, 2000.
4. Crolard, T. Subtractive logic, *Theoretical Computer Science*, 254(1):151–185, 2001.
5. Dubucs, J. Feasibility in logic, *Synthese*, 132(3):213–237, 2002.
6. Dubucs, J. and Marion, M. Radical anti-realism and substructural logicsm, [in:] Rojszczak, A., Cachro, J., and Kurczewski, G. (editors), *Philosophical Dimensions of Logic and Science*, pages 235–249. Kluwer Academic Publishers, 2003.
7. Dummett, M. Reply to Dag Prawitz, [in:] Taylor, B. (editor), *Michael Dummett: Contributions*

to Philosophy, pages 281–316. Distributors for the United States and Canada: Kluwer Academic Publishers, 1987.
8. Dummett, M. A. E. *The seas of language*. Oxford: Oxford University Press, 1993.
9. Dummett, M. A. E. *Elements of intuitionism*. Oxford: Oxford University Press, 2000.
10. Dunn, J. M. and Hardegree, G. *Algebraic methods in philosophical logic*. Oxford: Oxford University Press, 2001.
11. Goodman, N. The logic of contradictions, *Zeitschrift fur Mathematische Logic und Grundlagen der Arithmetik*, 27:119–126, 1981.
12. Hardegree, G. M. Completeness and super-valuations, *Journal of Philosophical Logic*, 34(1):81–95, 2005.
13. Heyting, A. *Intuitionism*. Amsterdam: North-Holland Pub. Co., 1971.
14. Kolmogorov, A. N. Zur Deutung der Intuitionistischen Logic, *Mathematische Zeitschrift*, 35:58–65, 1932.
15. Mancosu, P. *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*. Oxford: Oxford University Press, 1998.
16. Martin-Löf, P. On the meanings of the logical constants and the justifications of the logical laws, *Nordic Journal of Philosophical Logic*, 1(1):11–60, 1996.
17. Martino, E. and Usberti, G. Temporal and atemporal truth in intuitionistic mathematics, *Topoi*, 13(2):83–92, 1994.
18. Mortensen, C. *Inconsistent mathematics*, Vol. 312. Springer, 1995.
19. Nelson, D. Constructible falsity, *Journal of Symbolic Logic*, 14(1):16–26, 1949.
20. Pietz, A. *Not quite intuitionism*, 2011. http://epub.ub.uni-muenchen.de/12589/1/DGPhil_Pietz.pdf
21. Prawitz, D. Meaning approached via proofs, *Synthese*, 148(3):507–524, 2006.
22. Priest, G. Dualising intuitionictic negation, *Principia*, 13(2):165–184, 2009.
23. Raatikainen, P. Conceptions of truth in intuitionism, *History and Philosophy of Logic*, 25(2):131–145, 2004.
24. Rauszer, C. Applications of Kripke models to Heyting-Brouwer logic, *Studia Logica*, 36(1-2):61–71, 1977.
25. Ripley, D. Embedding denial, [in:] Caret, C., and Hjortland, O., (eds.), Foundations of Logical Consequence, Oxford University Press, forthcoming 2015.
26. Shramko, Y. Dual intuitionistic logic and a variety of negations: The logic of scientific research, *Studia Logica*, 80(2-3):347–367, 2005.
27. Shramko, Y., Dunn, J. M., and Takenaka, T. The trilaticce of constructive truth values, *Journal of Logic and Computation*, 11(1):761–788, 2001.
28. Skura, T. On pure refutation formulations of sentential logics, *Bulletin of the Section of Logic*, 19(3):102–107, 1990.
29. Słupecki, J., Bryll, G., and Wybraniec-Skardowska, U. Theory of rejected propositions. I, *Studia Logica*, 29(1):75–123, 1971.
30. Słupecki, J., Bryll, G., and Wybraniec-Skardowska, U. The theory of rejected propositions. II, *Studia Logica*, 30(1):97–145, 1972.
31. Solomon, G. and DeVidi, D. Empirical negation in intuitionistic logic, [in:] DeVidi, D., Kenyon, T. (eds.) *A logical Approach to Philosophy*, pages 151–168. Springer, 2006.
32. Tennant, N. Negation, absurdity and contrariety, [in:] Gabbay, D. M., Wansing, H., (eds.) *What is Negation?*, pages 199–222. Springer, 1999.
33. Trafford, J. Abstract duality and co-constructive logics, manuscript under submission, 2014.
34. Trafford, J. Duality and Inferential Semantics, *Axiomathes*, forthcoming 2015.
35. Urbas, I. Dual-intuitionistic logic, *Notre Dame Journal of Formal Logic*, 37(3):440–451, 1996.

Notes

1. To get ahead of ourselves, this property is mirrored in co-intuitionistic logic by the "conjunction property", where $\dashv \alpha \wedge \beta$ iff $\dashv \alpha$ or $\dashv \beta$, and, since we understand co-intuitionism constructively, the disjunction property can not be central to a definition of constructivity.
2. I will not enter into a historical exegesis regarding the two positions (on this, the references above are more than adequate).
3. "In addition to theoretical logic, which systematizes a proof schemata for theoretical truths, one can systematize a proof schemata for solutions to problems [...] In the second section, assuming the basic intuitionistic principles, intuitionistic logic is subjected to a critical study; it is thus shown that it must be replaced by the calculus of problems, since its objects are in reality problems, rather than theoretical propositions" [13, p.58] (Translated in [15]).
4. See also Martino and Usberti [17]:

   > From the fact that possibility is conceived as tenseless it follows that the following principle of Potential Excluded Middle: (PEM) A is potentially true or A is not potentially true becomes intelligible, and valid, in its classical reading. For, on this reading, it simply means that all propositions, as they are conceived by the potential intuitionist, are atemporally determinate, and this is clearly true: if it were indeterminate whether A is provable or not, the provability of A would be for ever prevented since, according to the conception at issue, a proposition cannot become provable. Therefore, such a hypothetical state of indeterminateness of A could be nothing but a state of well-determined unprovability of A. Whether A is provable or not is a fact concerning the immutable world of propositions, where there is no room for any indeterminateness.

5. Though formulas of S^d will typically be decorated with superscript in what follows (e.g. α^d) unless context is obvious.
6. Co-implication is sometimes called "subtraction", e.g. [11], and typically, "$\alpha \leftarrow \beta$" is read as "α without β", though this is not the case in our refutation interpretation. Further discussion can be found in [24], [35].
7. Obviously, we are working with a formal language here, so we probably don't strictly need to use the phrase "well-formed formula" (*wff*) of L. I should flag up that I interpret the phrase quite liberally, allowing, for example "$\{\phi \mid \ldots\}$" to be a *wff*.
8. See [34] for a discussion of the duality of classical logic in the context of constructing an inferentialist semantics.
9. This can be made formally precise through the notion of a Galois connection between two structures as explored in [33].
10. There are several issues regarding the interpretation discussed in [17, §11], that our interpretation in terms of refutation may solve, though this requires further investigation.
11. This is close to the view suggested in [35].
12. Note that this is a syntactic term, which must be distinguished from the semantic "indeterminate" of logics such as K_3.
13. One way around this (that I will not follow up here) might be to weaken the conditions on derivation of a contradiction, so that, for example, following [22], we think of $\perp$ as expressing something that is antithetical to the inquiry that we are interested in. So, we could think of $\perp$ as some theory that has no proof, but is much weaker than an outright contradiction. Then, any proof of $\alpha \Rightarrow \perp$ shows, indirectly that, α is antithetical to our inquiry at that stage. In other words, it allows us to express that, at that specific stage of reasoning, α is unprovable. Depending upon how this is cashed out, this would not necessarily preclude the existence of a proof of α at some later stage, in which, for example, our assumptions or data have changed such that α no longer leads us to a theory antithetical to inquiry.
14. This is close to Tennant's [32] conjecture (though clearly divergent from his conclusions), that negation may be understood as a constraint upon the space of possible proofs in terms of contrariness.
15. Though, this is not the route that I suggest.
16. If, for example, we relaxed the separability condition so that both negation operators may be applied to any formula, so that, $\neg_I \alpha^{\perp}$, and $\neg_C \alpha$ are *wff*'s, then an application of both negations to a formula, (e.g. $\neg_I \neg_C \alpha$), is obviously equivalent to Boolean negation. See also [4], [1].
17. We could, of course, add our "strong" negation along the lines of (3) and (4) given above, but I do not wish to confuse matters.
18. See e.g. [10], [12].

19. I borrow this term from [25], where it used in a rather different context but is conducive to this understanding: “a paraconsistent approach amounts to a paracoherent approach, in which one can be locally incoherent without global incoherence [...] Taking the paracoherentist option I'm suggesting here amounts to choosing to maintain a certain amount of reflective tension.”

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The Logic System is the Way You Do Logic

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Andrew Schumann: All the basic forms of modern communication in science have been generated in the early modern period, namely at that time the first scientific communities have been created and the first scientific journals have been established (so, the first scientific community was called *Academia secretorum naturae*, it was founded in Naples 1560, and the first scientific journal was called *Journal des sçavans* that started to be published 1665). In the sphere of symbolic logic you are an organizer of many communication platforms for scientists, e.g. you are the editor-in-chief or editor of many scientific journals, book series and most recently you have launched the publishing house, *College Publications*. Therefore the translation of your family name from Hebrew as the 'head of community' has a symbolic meaning. In logical communities you have organized and promoted high-grade scientific communications. How far traditional formats of scientific communication (submitting to journals, contributing to books, writing books, talking in scientific conferences), as continued since the early modern period, have been till now effective? What are the features of scientific communication in symbolic logic? Does it, perhaps, have the sense to invent absolutely new forms of scientific communication which did not previously exist? For example, an analogue of Wikipedia, i.e. a joint scientific source like Nicolas Bourbaki, but on the Internet?

Dov M. Gabbay: I think that publishing books, journals and so on, and publishing on the Internet allows us to get a lot of information very easily. But what has also happened is that the speed, in which we want to work and publish, has increased a lot, too. And the problem is that we face publishers who want to own scientific sources. I mean that two hundred years ago your publisher was paid for his job to print your paper and to promote it. Nowadays, publishers are handing over information, because they want to possess copyrights, so they don't give copyrights to scientists, i.e. authors. But the web developed very quickly. Sitting at home before writing a paper, you want to know, if your paper's possible contents is connected to another paper written by someone else. You go to the web and can find this other paper on a webpage of a journal. Then you have to pay about $35 just to take a look at it. And you don't know whether it will be suitable or not for your work. Hence, the situation is frustrating and will become worse.

Andrew Schumann: Maybe some additional questions about the same subject. For example, the speed of spreading some ideas (life-cycle of ideas) in mass-media, in show business is much faster than in scientific circles. Maybe we need some things that give us faster speed in the future in science too, maybe some kind of social networks, especially for logicians.

Dov M. Gabbay: Surely, we need to publish ourselves as widely as possible. But the main reason why it cannot be done at the moment, is because you need to be published in good high impact journals if you want to be promoted at your university. However, we don't have enough good open-access journals. I have, therefore, worked on a new journal, which I hope will be a good open-access journal published by *College Publications* http://www.collegepublications.co.uk/. Notice that this place is a good way to publish scientific monographs with very cheap prices. Recently, there is a movement against publishers because of their attitude to the business. It is a terrible thing – what is happening now. Publishers just want to earn money. They regard the scientific content as a product. They may as well be selling electric toasters. One day, I said to the senior executive of publishing house: 'You wear a suit, you look at spreadsheets and you calculate profits. That is all you care about and as far as you are concerned, you may be selling bagels.'

Andrew Schumann: You anticipate novel trends in symbolic logic, and have even established some. Which new trends in symbolic logic can we expect? What will spark the interest of logicians in the near future?

Dov M. Gabbay: There are things I've already said 20 years ago, but now there are people, who are working in the conventional way, regarding logical systems as theories, i.e. as something static. But I've said that a logical system should not be just what theories are, what you can infer from them, but also contain the way you get to these theories. If you want to construct proof theory by some algorithms, you know that if you change these algorithms even a little bit, you might get a different logical system. In other words, you know that a logical system is a system of algorithms. And to repeat what I said 20 years ago, is that the *logical system is a theory consisting of algorithms, but also containing various mechanisms like abduction, comparing, etc., which allowed us to add to this theory*. *And now I say that the logic system is not just the way it represents data and manipulate it but also contains algorithms of how you get more data and how you change it, evaluate it, label it. In other words it is also dynamics of reasoning, how you move about, how you think, how you execute your actions, part of the logic is also how you use it.* So, a logical system is really what is in the head of an agent and the way he is being treated relative to his environment. Logic is human: 'I am a logical system.' So, I think logic should go in that direction. I think logicians will get to this idea sooner or later but this process is very slow, because people are very conservative.

Andrew Schumann: Maybe especially logicians are very conservative.

Dov M. Gabbay: In order to be less conservative, we should work in many areas. For instance, I work in many areas and I see many connections. I see that in order to reconcile what's done here, what's done there and what's done in many other places you need a more general system. Take, for example, belief merging and voting. Voting is studied in social and political sciences and belief merging is studied in logic, but they are very similar processes. Nevertheless, it is what you want to do with in each area that is different. In both cases you have several candidates and you need to reconcile them. So, by looking at these all areas and things, we can guess, where the true facts of logic come from and where logic is going to go.

Andrew Schumann: For a couple of years now, you have done huge work on formalizing Talmudic logic. You have launched the book series, *Studies in Talmudic Logic* where the eleven books are already published. What is the general intention of this series? Which books have been published, which are planned? What is it, the logic of the Talmud? What features does it have?

Dov M. Gabbay: The books, which are already published, are written both in Hebrew and English. The Talmud is a set of rules and of arguments and it regulates how we behave. Therefore, it's merely modeling and guiding humans and so it has its own way of looking at what human beings do and what they can do with it. Therefore, if we model the way Talmudic scholars are thinking and the way they are arguing, we can get a new formal logic modelling the principles, which have been important since antiquity. In other words, this logic may cover our experiences of how to follow some temporal reasoning, action reasoning, etc. of the Talmud. In this way, we are forced to create new logics, new ideas and then we export them from the Talmud for modern logic. So, in many cases we have extrapolated new principles that were already in the Talmud, but they are new and not known in modern logic and artificial intelligence like Talmudic temporal logics and Talmudic cognition logics and we make a combination of those concepts and export to modern logic.

For example, the book we are working on now. We are investigating what to do if Rosh Hashanah is on Sabbath, this means that two bodies of laws apply to the same object. So, this is belief revision from the following point of view. We have different predicates A, B, C and then we come with an input, revision, saying A is the same as B. And it maybe that Y is true for A and Y is not true for B. However, if we say, A equals B, we have either Y or not Y, i.e. we have to choose between them. And the Talmud doesn't do it like this. The Talmud says: why is Y given for A and why is not-Y given for B? I mean, if we take Rosh Hashanah and Sabbath. On Sabbath we are not allowed to cook. On Rosh Hashanah we are allowed to cook. So, if Rosh Hashanah is on Sabbath are we allowed to cook or not? We don't know. So, we ask: why on Sabbath are we not allowed to cook? We say: cooking is work and Sabbath is the end of creation. So, the reason we don't work on Sabbath is because creation is over and G-d stopped working on creation. And cooking is work. Why on Rosh Hashanah can we or should we cook? It's because it's honouring and welcoming the New Year and we are supposed to be happy, so we are allowed to cook. And now, we understand the reasons for each rule and we can agree on their priorities. But actually, what the Talmud says is different. It says that on Rosh Hashanah, we are also not allowed to cook, but this rule is cancelled, because we are supposed to enjoy ourselves. So, really, if we put them together the determination is clear. We suspend the cancellation. Thus the Talmud uses a calculus of cancellations, a new concept applicable in modern artificial intelligence.

Thus, exporting new ideas and principles in the book series *Studies in Talmudic Logic* deals with many tasks. And using symbolic logic helps us. I believe that G-d created us as reasoning creatures in His own image. So, if we model Human logic, we get closer to G-d, and better understand His commandments. Let's take a painter. I don't know, which one you like. Perhaps Van Gogh? You see his paintings, so you can admire him by the painting, but not the person as such. However, if you could read the diaries and see the way Van Gogh was thinking we will have a better understanding of him as a person. So, in that similar way you may know how Talmudic logic works and understand another side of G-d. When G-d gave the Ten Commandments He also gave Moses the rules to derive more laws. These rules are closer to G-d's way of thinking.

Andrew Schumann: Why did you decide to be engaged in formalizing the Talmudic logic? What were the reasons: religious, philosophical, scientific? What do you like in Talmudic reasoning? Can we state that formalizing Talmudic reasoning will be useful in symbolic logic? In analytic philosophy? Can it find any application in computer science? Whether the logic hidden in the Talmud is a unified intelligent hyper-large system like *Solaris* of Stanisław Lem? Is it possible to say that this logic is the logic of the Lord?

Dov M. Gabbay: I wanted to do the Talmudic logic 50 years ago and I tried. There were many books on Talmudic logic written by Louis Jacobs and many others. But only just now the modern symbolic logic is ready to formalize the Talmudic way of reasoning. Only now after 50 years of developing many logical systems and working in many areas I felt ready that I could deal with it. The problem is that nobody has tried to formalize the Talmudic logic like we are doing in the *Studies in Talmudic Logic* book series. There are very great thinkers who know the Talmud, but they don't know the whole spread of logic. Suppose that you have been a plumber for 50 years, you would have a garage full of all kinds of things – cords, pipes, screws and screwdrivers, pieces of metals, washers, etc. So, if you see a problem you will immediately know what to do, because you can say: 'I got these little pieces of things from 30 years ago, I can use them here.'

There are very great traditions of Talmudic studies in Yeshibot (Rabbinic schools). And what I can do is to help the students of those schools to learn logical pieces of Talmudic reasoning. There are many people who simply want to study Talmudic logic, many of whom are young. Some students of Yeshibot ask me how many volumes are planned? I said I don't know. Maybe twenty or twenty four, maybe sixty or seventy. I don't know how far it will go.