Volume 3:1 (2014)

ISSN 2299-0518

Multi-disciplinary Quarterly Journal

STUDIA HUMANA

UNIVERSITY of INFORMATION TECHNOLOGY and MANAGEMENT in Rzeszow, POLAND

Editorial Office: University of Information Technology and Management in Rzeszow, Poland

2 Sucharskiego Street
Rzeszów 35 – 225

DE GRUYTER OPEN

studia humana
QUARTERLY JOURNAL

Studia Humana
Volume 3:1 (2014)

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studia humana
QUARTERLY JOURNAL

Studia Humana
Volume 3:1 (2014), pp. 3—12
DOI: 10.2478/sh-2014-0001

Gregory Palamas and Our Knowledge of God

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Abstract:
Although Gregory wrote very little about this, he acknowledged that natural reason can lead us from the orderliness of the physical world to the existence of God; in this, he followed the tradition of Athanasius and other Greek fathers. Unlike Aquinas, he did not seek to present the argument as deductive; in fact his argument is inductive, and of the same kind as – we now realize – scientists and historians use when they argue from phenomena to their explanatory cause. Gregory wrote hardly anything about how one could obtain knowledge of the truths of the Christian revelation by arguments from non-question-begging premises; but in his conversations with the Turks he showed that he believed that there are good arguments of this kind. Almost all of Gregory's writing about knowledge of God concerned how one could obtain this by direct access in prayer; this access, he held, is open especially to monks, but to a considerable degree also to all Christians who follow the divine commandments.[1]

Keywords: Gregory Palamas, inductive argument for God’s existence, Orthodox Christianity

This paper was first published in the volume
Schumann A. (ed), *Logic in Orthodox Christian Thinking.* Ontos Verlag, 2013, pp. 18–37.
In *Studia Humana* it is published by kind permission of Dr. Athanasopoulos.

Christian tradition has normally held that we may acquire knowledge of God by three routes-natural reason, publicly available revelation (contained primarily in Scripture), and individual direct awareness of God. In this paper I shall assess the views of Gregory Palamas on the nature and value of each of these routes.

Since Gregory was writing almost entirely for those who already believed Christian doctrines, he did not have much to say about our access to God by natural reason, and for that reason he has been viewed as denying the existence or importance of such access. That view of Gregory, I shall now argue, is mistaken. *Romans* 1:20 claims that

> ever since the creation of the world [God’s] eternal power and divine nature, invisible though they are, have been understood and seen through the things he has made.

Gregory was highly critical of Greek philosophy because he saw it as leading to polytheism;[2] he did not, I think, realize how disconnected were the religious practices of ordinary pre-Christian Greek people from the reasonings of some Greek philosophers. Nevertheless, like most Christian theologians, Gregory saw *Romans* 1:20 as telling us that non-Christians can learn by the exercise of their natural reason that there is a God of great power, knowledge and goodness who created and sustains the world. He wrote:

> knowledge of creation brought mankind to knowledge of God before the Law and the prophets; today also it is bringing men back; and almost the whole of the inhabited world... now possesses by that means alone a knowledge of God who is none other than the creator of this universe;[3]

and he claimed that by attending to the λόγοι of beings one comes to knowledge (γνῶσις) of 'the power, wisdom, and knowledge' of God (*Triads* II 3.15–16).

Barlaam however had pointed out that the rules of reasoning understood as the rules of a deductive argument, that is an argument which is such that to assert the premises but to deny the conclusion would be to contradict yourself, a syllogism (in a wide sense), had been codified by Aristotle; and that these had the consequence that there could be no apodictic syllogism (i.e. one with evident premises and so indubitable conclusion) which would demonstrate the existence of God from non-Scriptural premises (see [15], pp. 188–190). Barlaam gave various reasons for this. In particular the premises would have to be general metaphysical principles, which he calls 'common notions, hypotheses, and definitions,' ones involving concepts abstracted from sensibles. But Aristotle held that

> demonstrative knowledge must proceed from premises which are true, primary, immediate, better known than, prior to and causative of their conclusion (Aristotle, *Posterior Analytics*, 71b, 20–25).

These are, I think, excessively demanding conditions for demonstrative knowledge; but clearly no inference is going to be of any value unless its premises are better known than its conclusion. And, Gregory acknowledged, humans could not know 'common notions' well enough to demonstrate the existence of God. 'Common notions,' he writes 'depend on the intelligence of him who was last created,' ([13], Ep. I Ak 10) that is on mere human intelligence.

All of Thomas Aquinas's 'five ways' ([2], 1a. 2.3) to prove the existence of God invoke metaphysical principles of the kind which Barlaam must have had in mind, e.g. a premise of the first way is 'everything in the process of change is being changed by something else,' and a premise of the second way is 'a series of causes must stop somewhere.' These are not obvious truths, and that is why the five ways do not yield certainty. Nevertheless the subsequent Western medieval tradition from Scotus to Leibniz sought to give tight compelling deductive arguments which appealed to such general metaphysical principles, for the existence of God until it came in the nineteenth century to accept Kant's claim that this route would never yield certainty. It was not however characteristic of the patristic tradition to put natural theology into the form of a syllogism. Rather, the Fathers simply point to the facts of the existence of the universe or to its orderliness, and claim that these things are to be explained by the action of a benevolent creator. Although the Arabic philosophers (see the very thorough analysis of these arguments in [7]) discussed at length various versions of arguments from the mere existence of a physical universe, arguments which were later called 'cosmological arguments,' the brief discussions in the Greek Fathers concentrate more on arguments from the orderliness of the universe, producing versions of what were later called 'teleological arguments.'

The most sustained presentation of such an argument of which I know is that by Athanasius in sections 35 to 44 of *Against the Heathens*. He gives there many examples of the beneficent ordering of nature. Assuming that physical matter is of four kinds – earth, air, fire and water – he points out that, despite their contrary natures (earth and water move downwards, air and fire upwards), they are put together in such a way as to produce an environment in which humans can flourish. Thus:

> Who that sees the clouds supported in air, and the weight of the waters bound up in the clouds, can but perceive Him that binds them up and has ordered these things so? Or who that sees the earth, heaviest of all things by nature fixed upon the waters, and remaining unmoved upon what is by nature mobile, will fail to understand that there is One that has made and ordered it, even God? Who that sees the earth bringing forth fruits in due season, and the rains from heaven, and the flow of rivers, and springing up of wells, and the birth of animals from unlike parents, and that these things take place not at all times but at determinate seasons, – and in general, among things mutually unlike and contrary, the balanced and uniform order to which they conform, – can resist the inference that there is one Power which orders and administers them ordaining things well as it thinks fit? For left to themselves they could not subsist or ever be able to appear, on account of their mutual contrariety of nature ([5], *Against the Heathens* 36).

Similar but very brief arguments are to be found in Gregory of Nyssa,[4] Maximus,[5] and John of Damascus.[6] Both the latter also give a cosmological argument, indeed the one which seems to the source of Aquinas's first way, although not obviously in the form of a syllogism.

In the *Triads* Gregory also appeals to an argument of Athanasius's kind, though without any examples and in a passage which would be almost impossible to understand without any familiarity with simpler accounts of it:

> What man of reason who sees the evident differences between the essences of things, both the oppositions of their powers and the compensating origins of their motions, their incessant successions from contrary properties and the unmingled attraction from inconceivable strife, the conjunctions of separate and unmixable things in a unity which are spirits, souls, bodies, this harmony of things so numerous, this stability in their relations and positions, this conformity of states and orders to their essence, the indissolubility in their cohesion, what man taking all this into his mind, would not think of who had positioned everything so well in its place and established this admirable harmony among all things, and recognize God in his image and in the beings which derive their origin from him?[7]

It was, I presume, an argument of this kind which he called in his letter to Akyndinos a method by which thinkers ascend (ἀναβαίνειν) from creation to the Creator:

> For example, one can proceed from things which manifest goodness to goodness itself, and similarly with wisdom, providence, life, etc. In this manner one achieves a demonstration free from deceit (άφευδης ἀπόδειξις) that there exists one who is in all things and who is removed from and transcends all things, the many-named and unnameable super-essential essence ([13], vol. 1, *First Letter to Gregory Akindynos*, p. 216).

As Sinkewicz comments,

> there emerges from this letter a notion of demonstration quite distinct from that advocated by Barlaam and ultimately by Aristotle. It is a notion that seeks its justification not in the Greek philosophers but in the tradition of the Fathers ([15], p. 201).

Although Gregory hints that arguments to God may be *sui generis*, he and his predecessors are in fact giving an argument of a kind very familiar in science, history, and ordinary life, when we argue not – as concerned Aristotle – from cause to effect, but from effect to cause. Such arguments are not deductive, but (in a wide sense) inductive. They reach conclusions rendered probable by their premises, but not entailed by them.

Scientists argue from particular observations to some very wide theory which purports to explain the observations and also predicts much more; so the conclusion could be false even though the premises are true, but – if their inference satisfies certain criteria – the premises do make the conclusion probable. Neither Aristotle nor the medievals, East or West, had the slightest conception of the nature of inductive inference, and of the criteria which a cogent inductive inference needs to satisfy. My own account of these criteria is as follows.[8] I suggest that an argument from observed phenomena E to an explanatory cause H is cogent (i.e. renders its conclusion that H is the cause probable) insofar as (1) if H is true, it is probable that E will occur, (2) If H is false, it is improbable that E will occur, (3) H is simple. This pattern of argument is one much used in science, history, and all other fields of human inquiry. A detective, for example, finds various clues – witnesses reported seeing John near the scene of a burglary at the time when it was committed, John's fingerprints on a burgled safe, and John having the stolen money hidden in his house; and then claims that these clues make it very probable that John robbed the safe. This is because (1) if John did rob he safe it is quite probable that he would be seen near the burglary scene at the time the burglary was committed, that his fingerprints would be found on the safe, and that the money stolen would be found in his house; (2) if John did not rob the safe, it would not be probable that he would be seen near the scene of the burglary; and very improbable that his fingerprints would be found on the safe, and the money be found in his house; and (3) the hypothesis that John robbed the safe is much simpler than rival hypotheses which would satisfy criteria (1) and (2). John's defence lawyer could always suggest other possible explanations of the clues. He could suggest that Brown planted John's fingerprints on the safe, Smith dressed up to look like John at the scene of the crime, and without any collusion with the others Robinson hid the money in John's flat. This new hypothesis would lead us to expect the three clues just as well as does the hypothesis that John robbed the safe. But the latter hypothesis is rendered probable by the evidence whereas the former is not. And this is because the hypothesis that John robbed the safe is far simpler than this rival hypothesis. A hypothesis is simple, insofar as it postulates the existence and operation of few entities, few <u>kinds</u> of entities, with few <u>easily</u> describable properties behaving in mathematically <u>simple</u> kinds of way. The detective's hypothesis postulates one entity – John – doing one deed – robbing the safe – which makes it probable that the listed phenomena will occur; whereas the defence lawyer's rival hypothesis postulates three separate individuals acting without any collusion between them.

Scientists use this same pattern of argument to argue to the existence of unobservable entities as causes of the phenomena they observe. For example, at the beginning of the nineteenth century, scientists observed many varied phenomena of chemical interaction, such as that substances combine in fixed ratios by weight to form new substances (e.g. hydrogen and oxygen always form water in a ratio by weight of 1:8). They then claimed that these phenomena would be expected if there existed a hundred or so different kinds of atom, particles far too small to be seen, which combine and recombine in certain simple ways. In their turn physicists postulated electrons, protons, neutrons and other particles in order to account for the behaviour of the atoms, as well as for larger-scale observable phenomena; and now they postulate quarks in order to explain the behaviour of protons, neutrons and other particles. What they postulate makes probable the occurrence of the phenomena, which are otherwise not probable, and is simpler than rival explanations thereof because it involves the operation of far fewer kinds of entities behaving in mathematically simple ways.

I have argued at length over many years that the arguments of natural theology to the existence of God can be articulated in such a way as to exhibit the same pattern (see [19], [17]). I take as the phenomena requiring explanation first, the phenomenon of the conformity of all physical objects to laws of nature, which I understand as the phenomenon that they all behave in the same simple way (for example the law of gravity just is the phenomenon that all atoms attract all other atoms in accord with the same simple formula). Then secondly there is the phenomenon that these laws are such as to lead to the evolution of human bodies; and thirdly the phenomenon that human beings are conscious. I argue that to be a person a substance has to live for a period of time, to have some power (e.g. to move his body), some true beliefs, and some freedom to choose how to exercise his power. God is the simplest kind of person there could be – since there are no limits to the length of his life, his power, his true beliefs, and to his freedom of choice; he is eternal omnipotent, omniscient, and perfectly free. Omniscience entails knowledge of which actions are good, and perfect freedom involves freedom from influences deterring the agent from doing what he sees to be good. The good motivates; insofar as you recognize an action as good and can do it you will do it – unless subject to irrational influences. God, being perfectly free, is subject to no such influences; and so he will bring about what is good. He cannot bring about everything good; for whatever good universe he makes, a bigger one would be better. But humans have a unique kind of goodness, not possessed even by God: the ability to choose between good and evil. So is quite probable that God will bring about us, and so therefore the necessary conditions for our existence – an orderly universe in which our actions will have predictable effects, human bodies, and human conscious lives. But it is immensely improbable that there would be such a universe unless an agent made it, and God is by far the simplest such agent. So the general nature of the universe makes it probable that there is a God.

Now of course, all the Fathers from Athanasius to Gregory Palamas took for granted a totally erroneous physics, in assuming that all mundane substances are made of earth, air, fire, and water. But their main point was that the chemistry of substances is such that different elements fit together in such a way as produce an orderly world (of day and night, winter and summer, rain and sun, plants and animals) fitted for humans. And I too am arguing from the powers and liabilities of the elements, now known to be quarks, electrons etc., and their initial arrangements being such as to produce an orderly world. My basic point is the same as that of the Fathers, even if expressed in terms of modern physics, and articulated in a much more sophisticated and rigorous way than theirs.

The traditional objection to any argument to the existence of God, deductive or inductive, is that God is incomprehensible, so utterly different from anything mundane, that we cannot have any significant knowledge of what he is like. And a hard-line application of the *via negativa* would hold that all predicates ascribed to God either express what he is not (e.g. to say that he is 'immortal' is merely to say that he is not mortal) or what he causes in the world (e.g. to say that he is 'good' is to say that he causes a good universe); and Dionysius, much admired by both Barlaam and Gregory, does seem to say that (or almost that[9]), and so does Barlaam,[10] both Dionysius and Barlaam claiming that God is known through creatures only as transcendent cause. Aquinas discusses the view that all the positive predicates attributed to God are to be analyzed in this causal way, a view which he attributes among others to Moses Maimonides; and he rejects it. For since 'God is just as much the cause of bodies as he is of goodness in things' then

> If God "is good" means no more than that God is the cause of goodness in things, why should we not say 'God is a body' on the ground that he is the cause of bodies? [2], Ia.13.2.

And surely if all we could know about God is that he is something which causes the universe which is not bad, not weak etc., there would be no reason to worship him. He might be a powerful spider, or a being indifferent to human well-being, or some theorem of mathematics. We worship God because not merely is he the cause of goodness, but because he is perfectly good in himself and so loves his creation. And most of those who used the method of 'ascent' claimed in effect that not merely did it show that the universe had a cause, but that it showed something positive about that cause.

Aquinas claimed that natural reason can show us that God has whatever must belong to the first cause of all things, and he claimed to show that that included being one, simple, perfect, supremely good, limitless, omnipotent, unchangeable, eternal etc. These predicates, Aquinas claimed, do tell us what God is like, but they fail to represent it adequately. That is because 'God' is not in the same genus as 'human' (a point which, Barlaam claims, has the consequence that no syllogism can proceed from principles applicable to the created world to a conclusion applicable to God). These words, Aquinas claimed, are used analogically of God. The perfections such as goodness and knowledge which humans have to some degree exist in God *altiori modo* ([2], 1a.14.1 ad.1), and so we cannot grasp fully what they amount to. However, after this life the 'blessed,' Aquinas claimed, will actually 'see' the essence of God and not depend on natural reason for knowledge of it; and very occasional humans may see it even in the life ([2] 1a.12.1 and 2a2ae. 175.3). But no creature ever, Aquinas claimed, could 'comprehend' that essence ([2], 1a 12.7), that is understand it perfectly; even if a created mind can see what that essence is, it could never understand why it is like that. Aquinas had however a problem. God, he thought, was simple – but how could a simple thing have all these properties – omnipotence, omniscience etc. He solved this problem in a cavalier and superficially incomprehensible way, by asserting that really all these properties are the same property as each other, and the same as God himself! However, if we ignore this aspect of his view, what he was trying to say was: God is simple, we can know quite a bit about him, but we cannot know his deepest nature.

Now Gregory also thought: God is simple, we can know quite a bit about him, but we cannot know his deepest nature. But he put it differently, because he and Aquinas meant very different things by 'essence' (οὖσια, *essentia*). For Aquinas, the essence of a thing is whatever properties are necessary for the existence of a thing of that kind.[11] So of course omnipotence etc. belong to the essence of God. For Gregory the essence of a thing is its deepest nature, whatever underlies its other necessary properties. So, he reasonably claimed, we cannot know anything about God's essence. But we can know, he claimed, about God's greatness and power etc – things 'inseparable from God;' so he called them – following earlier writers – God's energies. And, following Basil,[12] he made the obvious point that these energies are distinct from each other; but since they do not belong to God's essence, that does not make God un-simple. So – just like Aquinas, Gregory held: God is simple, we can know quite a bit about him, but we can't know his deepest nature. But he expressed the point without needing to put it in Aquinas's paradoxical way.

I pass on to consider briefly Gregory's account of publicly revealed truth. This, he held, is provided by Scripture as interpreted by the Fathers. He certainly thinks that there are good deductive arguments from Scripture and from the Fathers, for truths of Christian doctrine.[13] Unfortunately however, as Gregory was well aware from his involvement in the controversies about the *filioque*, it is all too easy to derive contrary doctrines from verses of Scripture taken in isolation. The process of doctrinal definition must be a much more complicated one, consisting of interpreting some Biblical texts in the light of others which the Church saw as expressing already established doctrine, and in the light of knowledge provided by natural science, and allowing that some of the Fathers sometimes made mistakes. All of this was recognized by Augustine and Gregory of Nyssa.[14]

Further, Gregory seems largely have ignored in all his writing the issue of providing publicly available evidence in support of the claim that Scripture interpreted along the lines described above is publically revealed truth. In this he differs from the earliest fathers, such as Justin, Tertullian and Irenaeus who argued on historical grounds that the New Testament contained the teaching of the Apostles received from Christ, whose miracles, above all his Resurrection showed his divine status and so guaranteed the truth of his teaching. With the passing of time, public historical evidence about Christ and his teaching became less accessible, and then writers began to argue – albeit very briefly in comparison with the attention which they began to give to natural theology – that the very success of the Church (through the blood of the martyrs, and not the force of arms) and miracles associated therewith, showed that the Church founded by Christ had Christ's authority for its teaching.[15]

The systematic listing of a catalogue of kinds of evidence in favour of the truth of Christian doctrine by Duns Scotus at the beginning of his systematic theology, the *Ordinatio*, may have been untypical of medieval thinkers, but all the kinds of evidence he mentions were known to, and cited in an unsystematic way by, other writers; and Scotus himself quotes other writers, normally Augustine, who cite these kinds of evidence. Scotus lists ten separate reasons for the credibility of Holy Scripture, and thus of the doctrines which can be derived from it ([9], Prologue, 100–119): (1) *Praenuntiatio prophetica* (the fulfilment of Old Testament prophecy in the New); (2) *Scriptuarum Concordia* (scriptures have a common message, and that includes the common witness of the New Testament writers to the teaching and deeds of Jesus); (3) *Auctoritas Scribentium* (the human authors' conviction that they spoke with God's authority); (4) *Diligentia recipientium* (the careful way in which the Church formed the canon of scripture); (5) *Rationabilitas contentarum* (the intrinsic probability of its doctrines); (6) *Irrationabilitas errorum* (the inadequacy of objections to those doctrines); (7) *Ecclesiae stabilitas* (the long and constant witness of the Church); (8) *Miraculorum limpiditas* (Biblical and later miracles, including the great miracle of the conversion of the western world); (9) *Testimonia non fidelium* (alleged prophecies of pagan writers), and (10) *Promissorum efficacia* (the sanctifying power of the Church's teaching in the lives of the faithful). (1), (2), (3), (4), and (8) are all aspects of historical evidence for the miraculous foundation events of Christianity; (7), (8) and (10) involve the Church's fidelity to the teaching entrusted to it, confirmed by miracles; and its sanctifying efficacy; (5), (6), and (9) involve the prior probability of what was taught. Here we have, I believe, a cogent inductive argument for the truth of Christian doctrine which conforms to the criteria which I analyzed earlier, albeit one of a more complicated kind than an argument of natural theology. For it appeals to publicly accessible data which are best explained by supposing that God inspired the Church in its compilation of Scripture.

Scotus wrote some thirty years before Gregory's correspondence with Barlaam, but – as far as I know – there is no awareness of Scotus in the theological writings of Gregory Palamas. Gregory did however argue with Turks (as well as with Jews) during his captivity by the Turks. He resisted their suggestion that as they believe in his prophet, he ought to believe in theirs – on the ground that the Old Testament Scriptures which they also revered did not prophecy the advent of Mohammad, and that Mohammad's teaching, unlike that of Moses and Jesus, was not accompanied by miracles.[16] So, he was in effect appealing to Scotus's first and eighth criteria; and he clearly did think that there are publicly available reasons in defence of at least some aspects of Christian doctrine.

Gregory thought however that only someone who had learnt to converse with God could discourse with any certainty about God. To do the latter one needs to study the Scriptures and apply them, above all by prayer. It was the experience through prayer of the Church, and especially of the monastic community, which provides full justification of Christian belief. He vigorously opposed the view which Barlaam seemed to be advocating that wise Greeks ([14], Ep 1 Bar 22. 237.9–13), meditating on the eternal Platonic ideas, had attained a similar knowledge.

And that brings me to the view for which Gregory is best known: that humans in this life can have personal detailed awareness of God, that is of God's energies, not his essence. Sometimes Gregory writes as though this vision is to some extent available to many Christians: 'This knowledge (γνϖσις) beyond reason is common (κοινή) to all who have believed in Christ' (*Triads* II. 36) Yet elsewhere he suggests that only some Christians can obtain the vision: 'Those who have obtained spiritual and supernatural grace… becoming gods, in God they know God' (*Triads* II.iii. 68). But the fullness of this vision seems to be open only to monks, and indeed in writing to Balaam, Gregory denied that he himself had attained this vision; he had just smelled it from a distance and not yet grasped it.[17] But he adds that he has heard the testimony of fathers who have had this vision; the light of mount Tabor 'shines even now in the hearts of the faithful and perfect'

(*Triads* II. iii. 18). Someone who 'mysteriously possesses and sees this light… knows and possesses God in himself, no longer by analogy,' in contrast to one who 'possesses knowledge of creatures and from this by means of analogy... infers the existence of God' (*Triads* II. iii. 16). And the light of contemplation differs even from the light that comes from the Holy Scriptures, whose light may be compared to 'a lamp that shines in an obscure place;' whereas the light of mystical contemplation is compared to the star of the morning which shines in full daylight, that is to say to the sun' (*Triads* II.iii.18). Indeed this contemplation is not, 'unless the term is employed in an improper and equivocal sense' knowledge; but 'superior to all knowledge" (*Triads* II.iii.17). Although the way of impassibility is 'most appropriate for those detached from the world' (*Triads* II.ii. 20), those in the world must try to form themselves in accord with the divine commands, and that can change our 'changeable disposition' into a fixed and blessed state.

So in what sense is this contemplative vision 'superior to knowledge?' Since I have not myself had this 'vision,' and few others – according to Gregory – have had it in its fullness, I hesitate to try to make sense of the connection between this vision and knowledge proper, which – as he writes – must require 'images and analogies' (*Triads* I.iii.18). But there is a distinction very familiar to Anglo-American philosophers in a secular context between 'knowledge that' so-and-so, and knowing some person or thing, which may throw some light on what Gregory is saying. Gregory insists that the vision is available only to those who put Scripture into practice.[18]The hesychasts who know God do read the Scriptures; whereas, he claims against Barlaam, pagan philosophers have not had any participation in a spiritual and divine light.[19] Obviously, we can know a lot about someone, e.g. David Cameron, without knowing David Cameron personally. But I do not think we can know a person without knowing something about that person. I couldn't know David Cameron unless I could recognize him when I meet him; and that involves knowing something (indeed quite a lot) about him: that he looks like this, that I meet him often at a certain place, and that he thinks so-and-so. And plausibly the same goes for God. To know God, one has to know what one is looking for when one opens oneself to the spiritual world in prayer. Christian doctrine teaches one what God is like – for example loving (and the Scriptures tell us what God's love amounts to) and Trinitarian. That enables us to distinguish apparent awareness of other things (e.g. of oneness with nature, or of an evil demon) from awareness of God. It puts us in a position to recognize God, if he should show himself to us. And if one has practiced following the teaching of the Scriptures, one will be better aware of what God's commands mean; and perhaps also better suited to benefit from the vision of God, which otherwise might be overwhelming.

But why should we or even the monks themselves believe what Gregory says about this knowledge of God which the monks of Mount Athos believe that they have acquired? It is, I suggest, the most fundamental epistemic principle of all, which I call the Principle of Credulity, that it is rational to believe that things are as they seem to us to be – in the absence of counter-evidence (that is evidence suggesting that we are subject to an illusion.) If you believe that you are seeing an elephant in an English garden, you should believe that you are – in the absence of counter-evidence. In this case of course there will normally be some counter-evidence – other people tell you that elephants in England are always to be found in zoos or circuses. But nevertheless if things seem very strongly to be a certain way, it is rational to believe that things are that way, despite a significant amount of counter-evidence. If not merely do you seem to see the elephant, but see it from many angles, touch it and hear it, that experience will outweigh the contrary testimony; and it is then rational to believe that you are indeed seeing an elephant. So if you yourself are having overwhelming experiences apparently of God of the kind which Palamas describes, it is rational to believe that your experiences are veridical, whatever the counter-evidence, whatever the doubts expressed by others.

It is also a fundamental epistemic principle, the Principle of Memory, that it is rational to believe that we had the past experiences we seem to recall – in the absence of counter-evidence (for example evidence that the thing recalled is very unlikely to have happened). And it is a third fundamental epistemic principle, the Principle of Testimony, that we should believe what other people tell us about their experiences – in the absence of counter-evidence (for example evidence that they are unreliable witnesses). And whenever there is counter-evidence which is strong enough to show that it is not rational to believe some apparent experience, memory, or testimony, the force of that counter-evidence can itself be defeated by counter-counter-evidence in the form of evidence showing that the counter-evidence was unreliable or additional evidence in favour of the truth of the original claim. In the elephant example, counter-counter-evidence to the belief that you are seeing an elephant might be provided by reading in the newspaper that an elephant has escaped from a local zoo, which would make it again rational to believe that you are seeing an elephant in an English garden, despite the counter-evidence that people tell you that in England elephants are always to be found in zoos or circuses.

People write books and articles for which they feel there is a need. And Gregory rightly did not think that there was a great need either for natural theology or for an impartial justification of Christian doctrine among the fourteenth century Greeks to whom he ministered. And so it is understandable that he did not write much about these first two routes to knowledge of God. We however in twenty first century Europe are surrounded by people who need these things, and I have been justifying the view that Gregory would have been sympathetic to the approach to them to which I have devoted most of this paper and which I have been commending. But Gregory did of course think that there was a great need in the fourteenth century for the direct awareness of God which comes through prayer; and who could doubt that the same applies today?

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Notes

1. The paper was read at the International Conference on St Gregory Palamas (Thessaloniki March 7–15, 2012) and it is published with the permission of Dr. C. Athanasopoulos, Editor of the Proceedings of the Conference.
2. ‘By examining the nature of sensible things [Greek philosophers] have arrived at a certain concept of God, but not at a conception truly worthy of him and appropriate to his blessed nature… For if a worthy conception of God could be attained through the use of intellection, how could these people have taken the demons for gods, and how could they have believed the demons when they taught men polytheism’ – [14], *Triads* 1.1.18. (All citations from *Triads* are from [14] unless otherwise stated.)
3. Ἐπέστρεφε τοίνυν ἡ τῶν κτισμάτων γνῶσις πρὸς θεογνωσίαν τὸ γένος τῶν ἀνθρώπων πρὸ νόμου τε καὶ προφητῶν, καὶ νῦν αὖθις ἐπιστρέφει, καὶ σχεδὸν πᾶν τὸ πλήρωμα τῆς οἰκουμένης, ὅσοι μὴ τοῖς εὐαγγελικοῖς θεσπίσμασιν εἴκουσι, δι᾽ αὐτῆς μόνης, οὐχ ἕτερον ἀρτίως ἔχουσι Θεόν, ὅτι μὴ τὸν ποιητὴν τοῦδε τοῦ παντός. – [13], *Triads* 2.3.44. The bold claim that theism is becoming universal seems to involve a favourable reference to the growth of Islam.
4. Gregory of Nyssa ch.1.
5. Maximus, 10.35–36.
6. John of Damascus, 1.3.
7. Τίς γὰρ νοῦν ἔχων καὶ ἰδὼν ἐμφανεῖς μὲν οὐσιῶν διαφορὰς τοσαύτας, ἀφανῶν τε δυνάμεων ἐναντιότητας καὶ ἀντιρρόπους κινήσεων ὁρμάς, ἔτι δὲ στάσιν τρόπον ἕτερον ἀντίρροπον, διαδοχάς τε ἀνεκλείπτους ἐξ ἐναντιοπαθείας καὶ φιλίαν ἀσύγχυτον ἐξ ἀσυμβάτου νείκους, συνοχάς τε τῶν διακεκριμένων καὶ ἀσυμμιξίας τῶν ἡνωμένων, νῶν, ψυχῶν, σωμάτων, τὴν διὰ τοσούτων ἁρμονίαν, τὰς μονίμους σχέσεις τε καὶ θέσεις, τὰς οὐσιωδεις ἕξεις τε καὶ τάξεις, τὸ ἀδιάλυτον τῆς συνοχῆς, τίς τὰ τοιαῦτα πάντα ἐπὶ νοῦν λαβὼν τὸν ἐν ἑαυτῷ ἕκαστον καλῶς ἱδρύσαντα καὶ πρὸς ἄλληλα θαυμασίως ἁρμοσάμενον οὐκ ἐννοήσειεν, ὡς ἀπ᾽ εἰκόνος καὶ αἰτιατοῦ γινώσκειν τὸν Θεόν [13], *Triads* 2.3.44.
8. I summarize here an account given fairly briefly in [19], ch 2, more fully in [17], chs 2 and 3, and yet more fully in [16], ch 4.
9. Dionysius writes that ‘the way of negation appears to be more suitable to the realms of the divine’ and ‘positive affirmations are always unfitting to the hiddenness of the inexpressible’ [8], *Celestial Hierarchy* 2.3. However, Dionysius claims, God has the ‘positive names of everything that is ... for he is their cause, their source and their destiny’ [8], *Divine Names* 1.7. So Scripture uses for God ‘names drawn from all the things caused: good, beautiful, wise, beloved…’ (op. cit. 1.6). Nevertheless ‘the unnamed goodness [that is, God] is not just the cause of cohesion, or life, or perfection, so that it is from this or that providential gesture that it earns a name, but it actually contains everything beforehand within itself.’ (op. cit. 1.7).
10. See the extract from Barlaam’s first letter to Palamas cited in [15], n. 169.
11. ‘Essence (*essentia*) or nature (*natura*) includes only what defines the species of a thing’ [2], Ia.3.3.
12. ‘The energies are various, and the essence simple’ (Basil of Caesarea, *Epistle* 234.1).
13. [14], Ep. 1 Ak 10 214.18–215.2.
14. For my account of how Scripture should be interpreted, derived from their teaching, see [18], ch 10.
15. See for example [1], 1.6. entitled ‘That to give assent to the truths of faith is not foolishness even though they are above reason.’ Aquinas claims that the divine wisdom ‘reveals its own presence, as well as the truth of its teaching and inspiration, by fitting arguments; and in order to confirm those truths which exceed natural knowledge, it gives visible manifestation to works that surpass the ability of all nature.’ So ‘above reason’ must mean merely ‘not susceptible of demonstration by an apodictic syllogism.’
16. See the analysis of Palamas’s own account of these controversies in [4], pp.104–18.
17. [14], Ep. 1. Bar. §10.230.6–11. Using the analogy of the vision to honey, Gregory writes that he is running towards the smell of honey but has not grasped it in his hands.
18. ‘Let us seek how to seek this glory and see it. How? By keeping the divine commandments’ (*Triads* II.iii.16).
19. ‘The light that shines even now in the hearts of the faithful and perfect… has nothing to do with that which comes from Hellenic studies, which is not worthy to be called light’ (*Triads* II.iii.18).

Studia Humana
Volume 3:1 (2014), pp. 13—26
DOI: 10.2478/sh-2014-0002

Philosophical Problems of Foundations of Logic

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Abstract:
In the paper the following questions are discussed: (i) What is logical consequence? (ii) What are logical constants (operations)? (iii) What is a logical system? (iv) What is logical pluralism? (v) What is logic? In the conclusion, the main tendencies of development of modern logic are pointed out.

Keywords: modern logic, criticisms of logic, normative logic, abstract logic, universal logic, combining logic

This work is supported by Russian Foundation for Humanities, grant No. 11-03-00761.

1. I want to bring into consideration the main logic development trends at the end of 20th century and the beginning of the 21st century. In the same way that the problem of the foundations of mathematics has risen hundred years ago, now there is the problem of the foundations of logic.

The following sections refer to:

(i) What is logical consequence?
(ii) What are logical constants (operations)?
(iii) What is a logical system?
(iv) What is logical pluralism?
(v) What is logic?

Each of the above-mentioned problems has been discussed intensively for recent decades, and there is plenty of literature on each issue. I fragmentary discussed some of these issues in [67], [68], and [69].

2. The standard definition of a subject of logic is the following: the science which studies *the principles of correct reasoning*. However, such a definition does not solve the problem of exact area of the given subject, i.e. what is the area of application of logic? For traditional logic, it is *syllogistic* reasonings, and there are 24 equally correct syllogisms. The nature of reasonings can vary greatly. For example, mathematical logic studies *mathematical* reasonings: "If […] his researches are devoted first of all to study of mathematical reasonings, the subject of his investigations can be called mathematical logic" (see [90]). In turn, *fuzzy* logic studies fuzzy reasonings, i.e. it deals with reasoning that is approximate rather than fixed and exact ([42]), informal logic studies *informal* reasonings (see [48]), and philosophical logic, as a result, studies *philosophical* reasonings. Then psychologistic reasonings are studied by *psychologistic* logic… In order to avoid similar senselessness, it is necessary to select the nucleus or base concepts with which the given science deals.

3. Such a nucleus undoubtedly is the concept of '*logical consequence*'. Logical consequence is the relation between premises and conclusion of a valid reasoning. Alfred Tarski in 1936, as one of creators of modern logic, sketched its essence in the work with the characteristic title "On the Concept of Logical Consequence" (see [119])[1]. However, we can add there the methodological aspects: in what terms it is defined or what paradigm of the offered answer is. Approaches to the answer concerning the sphere of application of logic, its basic concepts, which are used by the conception of logical consequence, may be completely different: model-theoretic, semantic set-theoretic, proof-theoretic, constructive, combinatory, etc. As we shall see, Tarski's answer is within the framework of the semantic approach:

> "The sentence *X follows logically* from the sentences of the class *K* if and only if every model of the class *K* is also a model of the sentence *X*" [119, p. 417].

Nowadays Tarski's concept of logical consequence is regarded as debatable. Tarski's work has more philosophical, nontechnical character and allows to interpret it in various conflicting ways, for example, there is an opinion that Tarski's definition is incorrect from the point of view of modern mathematical logic (see [24]) or that it should be generally rejected (see [30]). Van McGee in [86] has continued this attack. An interesting analysis of Tarski's work is proposed in [102], where he examines three basic concepts of logical inference, each of them envelops an important part of argument and each of them is accepted by logical community. The interesting conclusion of the author is that Tarski does not tell, what the logical consequence is, but considers what the logical consequence is similar to. Ray in [98] and Sher in [110] has defended Tarski's analysis against Etchemendy's criticisms in his big article (see the reply in [4] and [51]. Of particular interest is the article of Gómes-Torente (see [45] where the author discusses, analyses and defends from a historical perspective some of the aspects of Tarski's definition of logical consequence. As noted by Shapiro in [106, pp. 132, 148]: "There have been, and still are, a variety of characterizations of intuitive idea that a sentence (or proposition) Φ is a logical consequence of a set Γ of sentences (or propositions)", and he leads not an exhaustive list of definitions (with his ten definitions), beginning with Aristotle.

The debate continued in the 21st century (see [14], [111], [28], [88], [5], [108], [63], [77], [78], [31],[47], [89], [25], [101], and [104]). In the last work the author rejects the standard definition of logical consequence and suggests a sufficiently general form of the consequence relation between *abstract signs*.

The basic objections against Tarski's definition of the concept of logical consequence are as follows. Anywhere in [119] it is not stipulated that the data domain should vary, as it is in modern logic (see [24, p. 43]). Logical properties, in particular the general validity of the argument of logical consequence, should be independent of a separately selected universal set of reasonings, in which language appears interpreted. Otherwise, many statements about a cardinality of data domain at a special interpretation of language can be expressed only by means of logical constants and, as result, they should appear logically true. However, Tarski himself considers the idea of the term 'logic' as excluding from the logical truths any statements about a cardinality, let even of logical area. Another objection is directed against Tarski's acceptance of the ω-rule (the rule of infinite induction) at formalizing first-order arithmetic. However, actually it was only a version of this rule in the simple theory of types. In connection with these objections it is necessary to make some general notes. Tarski knew very well Gödel's works about the completeness, where the theorem is proved on the basis trueness of statements at all possible interpretations, as well as about the incompleteness (ω-incompleteness) of first-order arithmetic. In the first case one showed a concurrence of logical consequence in the first-order classical logic with syntactic consequence, in the second case one did not. From Tarski's works it clearly follows that he considers the logical consequence and deductability as different concepts and the first as much wider one than the second. The basic intention of Tarski was to define the logical inference, applicable to very wide class of languages, so wide that, as we shall see further, there are problems of the whole other level relating to the question ‘What is logic’.

For now note that the concept of logical consequence has taken the central place in logic and therefore the following problem seems to be very important: *What does this mean for the conclusion A to be inferred from premises Γ*? The following criterion is considered conventional: *A* follows from premises *Γ* if and only if any case, when each premise in *Γ* is true, is the case, when *A* is true. Significantly, the famous Russian logician Andrey Markov (the founder of constructive mathematics in USSR) connects this principle to the definition of what logic is: “Logic can be defined as a science about good methods of reasoning. By “good” methods of reasoning it is possible to mean ones, where from true premises we infer a true conclusion” (see [82, p. 5]). As a result, the essence of logical consequence is preservation *of truth* in all cases. There are many ways, when, using Tarski’s concept of logical consequence, it is possible to represent all laws of classical logic as valid. Thus, we obtain a standard definition of this logic together with all its logical operations. For instance, the conjunction of two formulas $A \wedge B$ is true at a situation (in a possible world) *w* if and only if *A* is true in *w* and *B* is true in *w*.

But we have much more problems there. Why do we call the obtained logic classical and what does this mean? We still consider this problem. What does ‘the standard setting of truth conditions for logical connectives’ mean? Finally, what should we consider as logical constants (operations)? The concept of truth is directly connected to the understanding of logical consequence, given by Tarski, and altogether results in objects which we call ‘logical laws’: *they are deductions preserving the truth*. But how can we define the logical law, not having defined what we should consider as logical constants, while we have a natural variability and instability of non-logical objects of reality. If we consider all objects as logical terms: variables, numbers, etc., then a model-theoretic interpretation of each term should be fixed and, therefore, only one model should exist. It would make the concept of logical truth empty.

4. Tarski in the end of his paper notes that the definition of the notion of logical consequence strictly depends on the distinction between logical and extra-logical constants. Because, if all the primitive terms are counted as logical constants, then logical consequence collapses into material consequence: *A* is consequence of *Γ* if and only if either *A* is true or at least one member of *Γ* is false. On the other hand, me must include the implication sign or the universal quantifier among the logical constants, otherwise “our definition of the concept of consequence would lead to results which obviously contradict ordinary usage” (see [119, p. 418]. Tarski writes that he does not know any objective basis for strict differentiation of these two groups of terms, and he concludes that this distinction between logical and extra-logical constants is the next big unsolved problem.

It is obvious that this problem did not give rest him and in thirty years he comes back to it in the lecture “What are logical notions?” read in 1966 in London Bedford College, in the same year in the Tbilisi Computer Center, and later in SUNY, Buffalo in 1973 (published posthumously in [120]). Tarski extends an area of discourse of applying Klein’s *Erlanger Program*, where one proposed a classification of various geometries in accordance with the space transformation, when geometrical concepts are invariant. For example, concepts of Euclid’s metric geometry are invariant relatively to isometric transformations. In the same way, algebra can be considered as study of concepts, invariant relatively to automorphisms of such structures as rings, fields, etc. The basic idea consists in that logical notions, i.e. sets, classes of sets, classes of classes of sets, etc., quantifiers, truth functions (implication, conjunction, disjunction, negation, etc.), should be “invariant under all possible one-one transformations of the world onto itself” (see [120, 149]). In other words, Tarski identifies logical notions with those notions that are invariant under all permutations of the universe of discourse (data domain). A similar idea had been previously maintained in [83]. Lindenbaum and Tarski in [73] showed that all logical notions from *Principia Mathematica* are invariant in this sense.

In one form or another an idea of an invariant permutability was discussed in various works in mathematical and philosophical logic (see [91], [92], [93], [84], [85], [114], [8], [109], [112], [87], [105], [32], [34], [125], [46], [7], [59], [21], [15], [76], and [16]). In the last work the authors come from the close connection between logical constants and logical consequence, and they investigate a function extracting the constants of a given consequence relation.

In [109, p. 53] is given a characterization of logical constants relatively isomorphic invariance which is a generalization of Tarski's approach. In the important work (see [87]), where criterion for logicality is invariance under bijections across universes, it is shown that if Tarski's thesis is accepted, then logical operations are defined in the full infinitary language $L_{\infty,\infty}$. Recall that the language $L_{\infty,\infty}$ is a language of conventional first-order logic (FOL) with equality (Frege's language), but admits conjunctions and disjunctions of an arbitrary length and as well as an arbitrary length of sequence of universal and existential quantifiers. This language is very rich – it contains the whole second-order logic (SOL), which is the extension of FOL by allowing quantifiers not just over individuals in the domain of discourse, but also over subsets of that domain and over relations and functions on the domain. Not only arithmetic, but also a set theory are included in SOL (natural numbers, sets, functions, etc. are there logical notions), as a result, all set-theoretic problematics, including the continuum hypothesis and many other important mathematical statements, are contained in SOL (see [81]). Thus, *mathematics is a part of logic*. Depending on expressive means of new logic, we come to logic of natural numbers, logic of real numbers, logic of topological spaces, etc. In the end, McGee accepted the Tarski-Sher thesis as a necessary condition for an operation across domains to count as logical, but not a sufficient one.

In connection with these problems Feferman's article [32] seems to be very interesting. In this article Feferman criticizes McGee's proposal and one of objections is that there is an assimilation of mathematics by logic. But the main objection is the following: "No natural explanation is given by it of what constitutes the same logical operation over arbitrary basic domains" (p. 37). The solution is to introduce invariance under mappings ("homomorphism invariance") instead of invariance under bijections. Such operations, according to Feferman, are logical and, it is the most remarkable, they exactly coincide with operations of the first-order logic without equality. However, here again there is a problem whether the equality may be considered as a logical operation? See the discussion of this problem in [96, pp. 61 ff], where Quine leans toward the positive answer. As a value of his approach, Feferman considers that the operations of FOL are defined in terms of homomorphic invariant operations of one-place type. Thus, he refers to [69], where the central role of one-place predicates in human thinking is shown by the example of the natural language.

Continuation of Feferman's ideas is the article [21], where the author characterizes the invariant operations as definable in a fragment of FOL. According to his notion of invariance, negation, arbitrary conjunctions and universal quantification are not invariant. As Casanovas notices, "… it is not easy to accept that universal quantification and conjunction are less logical than existential quantification and disjunction" (p. 37). On the other hand, it follows from his results that some particular forms of equality are invariant. Casanovas' work makes you think seriously about the criteria of invariance.

Now it is clear that the characterization of logical operations entails the characterization of the logic as a whole.

5. Note that the characterization of FOL can be given in terms of fundamental model-theoretic properties of the theory T in the first-order language. These properties are:

The compactness theorem (for countable languages). *If each finite set of propositions in T has a model*, *then T has a model*.

The compactness takes place, as only the finite set of premisses is used in deductions. This property was revealed by Kurt Gödel in 1930 in his paper about the completeness of FOL. One consequence of compactness is what is often called the upwards Löwenheim-Skolem theorem: *If T has an infinite model, then T has an uncountable model*.

The following property of FOL was proved earlier.

The (downward) **Löwenheim–Skolem theorem**. *If T has a model, then T has a countable model, too*.

Much later Lindström in [74] showed that these properties are characteristic for FOL in the following sense:

Lindström's theorem. The first-order logic is a maximal logic (closed under ∧, ¬, ∃) which satisfies the compactness theorem and the (downward) Löwenheim-Skolem's theorem.

Lindström's paper became paradigmatic for the major researches in logic of the last quarter of the 20th century. In essence, Lindström's theorem defines FOL, more precisely FOL(=), in terms of its global properties. But a serious limitation on expressive means of FOL follows from these properties. The simplest infinite mathematical structure is constituted by natural numbers and the most fundamental mathematical concept is the concept of finiteness. However, from the theorem of compactness it follows that central concepts such as finiteness, countability, well-orderedness, etc. cannot be defined in first-order logic. Actually, the finiteness is not distinctive from the infiniteness. In turn, from Löwenheim-Skolem's theorems it follows that the first-order logic does not distinguish the countability from the uncountability and, hence, no infinite structure can be described up to isomorphism. Moreover, many linguistic concepts, distinctions and constructions are beyond applications of FOL (see [43][2] and [75]). Of course, FOL possesses such attractive properties as *soundness* (a soundness property assures us that a formal system is consistent) and *completeness* (a formal system is 'semantically' complete when all its valid formulas are theorems), but our knowledge is often inconsistent, incomplete, and nonmonotonic[3].

There is a lot of interesting logics, which are richer than the first-order logic such as the weak logic of the second order which tries to construct the concept of finiteness in logic in the natural way (it allows to quantify over finite sets); logics with various extra-quantifiers such as there exists finitely many', 'there exists infinitely many', 'majority', etc.; logics with formulas of infinite length; logics of the higher-order (see [11]). However, it doesn't matter how we extend FOL – in any case we lose either the property of compactness, or Löwenheim-Skolem's property, or both as well as we lose the interpolation property and in most cases completeness. However, Boolos (see [17]) protecting the second-order logic, asks: Why the logic should necessarily have the property of compactness? It is interesting that we find a similar question in 1994 on pages of 'The New Encyclopedia Britannica': Why Löwenheim-Skolem's property should correspond to the internal nature of logic? (Vol. 23, p. 250). In [99, p. 304] the author argues that "the lack of completeness theorem, despite being an interesting result, cannot be held against the status of SOL as a proper logic."

The construction of various extensions of FOL, especially logics with the generalized quantifiers, drew big attention of linguists, mathematicians, philosophers, cognitivists. A total of development of this direction is reflected in the fundamental work 'Model-Theoretic Logics' (see [2]), where Barwise comes to the following conclusion: "Mathematicians often lose patience with logic simply because so many notions from mathematics lie outside the scope of first-order logic, and they have been told that that *is* logic […] There is no going back to the view that logic is first-order logic." (p. 23). Shapiro in [107] is of the same opinion too. His book presents a formal development of second- and higher-order logic and an extended argument that higher-order formal systems have an important role to play in the philosophy and foundations of mathematics.

However, SOL is too complicated. Incompleteness of SOL means that this formal system does not properly capture logical consequence. The basic problems arise with logical truths. For example, there are statements which are logically true if and only if the generalized continuum hypothesis holds. All these difficulties and many other are an inevitable corollary of a huge potency of expressive means of second-order languages.

Probably, one of the most interesting extensions of FOL belongs to Hintikka [54]. He enveloped *independence friendly logic* (IF logic) which is an extension of FOL with existential quantifiers $\exists x/y$, meaning that a value for x is chosen independently of what has been chosen for y. IF logic has the same expressive power as existential second-order logic. Although IF logic shares a number of metalogical properties with FOL (among them Lindström's theorem), there are some important differences. Due to its greater expressive power, IF logic is not axiomatizable. It means that IF logic is semantically incomplete. On the other hand, IF logic admits a self-applied truth-predicate and possesses many other interesting properties (see [123], [80]. Pay attention to the papers with the title 'A Revolution in Logic?' (see [57]) and 'A Revolution in the Foundations of Mathematics?' (see [55]). However, Hintikka's proposal that IF logic and its extended version be used as new foundation of mathematics has been met with skepticism by some mathematicians, including Feferman [33].

6. Apparently, we should agree with Benthem and Doets (see [11, p. 235]) that "No specific theory is sacrosanct in contemporary logic." This point of view, the authors add, applies also to alternatives to classical logic (such as intuitionistic logic). In general, it is possible to consider it as the answer to Tharp's article 'Which logic is the right logic?" (see [122]).

It's worth stressing that the traditional approach to the understanding what logic is seems to be very attractive in respect to the possibility to define logic by means of its basic laws. As Frege wrote in 1893: "Laws of logic … are the most general laws, which prescribe universally the way in which one ought to think if one is to think at all" (see [38, p. 12]). Then one of the philosophical problem in foundation of logic is the critics of basic logical laws undertaken early in the beginning of the twentieth century by L. Brower (Law of the Excluded Middle), Vasilyev and Łukasiewicz (Law of Non-Contradiction). Different systems of intuitionistic and paraconsistent logics first appeared as the result of this process. Later Lewis criticized the main properties of material (classical) implication in 1912, and Ackermann rejected the properties of strict implication in 1956. Thus multiple systems of modal and relevant logic appeared. Subsequently, criticism of basic logical laws became total, and it is worth to say that by the 20th century none of the ever known classical laws remained undoubted. Even the implicational law of identity $A \to A$ does not bear the test of time. Since, according to E. Schroedinger, generally it has no place for microscopic objects. Such logics are called 'Schrödinger logics' (see [26]).

Eventually this led to the extreme diversity of non-classical directions in logic (see [40][4], ([49], [117], and [94]). Unexpected result of this process was the appearance of huge classes of new logical systems. Moreover, in the most cases cardinality of these classes equals to continuum. The first outcome of a similar sort belongs to [64] and concerns a cardinality of the class of extensions of intuitionistic logic. Also there are continual classes of Lewis' modal systems, relevant systems, paraconsistent systems and so on. Now the discovery of the continual classes of logics is the most ordinary thing (see [44]). In this work it is shown how continual families of logics are *normally* built and what corollaries can be obtained from the corresponding construction.

Recently discussion about the nature of logical consequence and the view that there is more than one 'correct' conception of logical consequence has given new impetus to the development of the idea of 'logical pluralism' (see [6]). In the paper [35] the attempt is made to maximally limit the scope of logical pluralism. As noted in [100], "historical discussions have usually presupposed that if one of the logics is correct, then that it is correct for all and everyone".

The unusual variety of logical systems and logical tools for proving theorems, the possibility of representation of the same system in different ways (Gilbert's style, natural deduction, sequent calculus, analytic tableaux, etc.)[5], and the fact that logic becomes more vital in the computer science, artificial intelligence, and programming led to the publication of the collected works (in England and in one year in the USA) with the title 'What is a logical system?' (see [39]). Generally speaking, the problem is formulated as follows: whether there is the one "true" logic and in the case if not, how we can limit our notion of logic or, more precisely, of a logical system?

In fact, everything looks much more difficult. On one hand, deadly criticism of "basic" laws of classical logic, on the other hand, almost unlimited extensions of the concept of the logical truth (in essence this process is inverse to the first), various specifications of the concept of logical consequence, and the same is about logical notions, evident inadequacy of formal-logical constructs in relation to the way in which the actual process of human reasoning takes place, serious problems (hardly explainable) that appear intuition of logic (see [126], [79]), the development of computer science and artificial intelligence – all of that points at the *global crisis* in the foundations of logic and clearly raises the question '***What is logic?***'

7. Exactly in hundred years after the appearance of Frege's well-known work '*Begriffsschrift*' (see [37]), in which predicates, negation, conditional, and quantifiers are introduced as the basis of logic, and also the idea of formal system is introduced, in which demonstration should be carried out by means of obviously formulated syntactic rules, – after hundred years of the triumphal development of logic as the independent science calling the worship, surprise, and occasionally bitter dismissal and even revenge for its former adherents and the mystical fear for the majority of others, suddenly there is Hacking's article under the title 'What is a logic?' (see [50]). Hacking highly evaluates Gentzen's introduction of structural rules, because the operation with them allows us to express the aspects of logical systems in which the role of constants is entirely given by their elimination and introduction rules, without any appeal to semantic notions. This important discovery is made by Gentzen in 1934. The presentation and development of logic by the way of sequent calculus, where the principles of deduction are set by the rules, permitting to pass from one statements about the deducibility to others, allowed Hacking to define logic as science about deduction. Therefore there are some reasons why Hacking's article is in the beginning of the above mentioned collected works [39].

Let us note that under the same title as Hacking's paper the works by several outstanding logicians have emerged (see Wang [124], Hodges [61], Hintikka and Sandu [58]). In these papers is gathered the big amount of historical, factual and analytical material concerning the great science aspiring to study the principles of correct reasoning. Of course, it is necessary to discuss the sphere of application and the limits of logic (see [65], [105], [127], [62], [56].

Not many working, qualified logicians think that logic is related to the laws of thought. In the second edition of HPL Hodges expands his paper devoted to elementary predicate logic from the first edition of HPL with the section 'Laws of Thought?', at the beginning of which he writes: "The question whether the sequent $p \wedge q \models q$ is valid has nothing more to do with mind than it has to do with the virginity of Artemis or the war in Indonesia" (see [60 p. 100]). Complete disappointment with the current state of logic is expressed in [9], when Benthem writes about himself: "who has taken a vow to study methods per se, chastely staying away from the wear and tear of the realities of reasoning." The prospects of the development of logic are also sketched there. Although, most logicians would agree with Van Benthem "if logical theory were *totally disjoint* from actual reasoning, it would be no use at all" [10, p. 69]. The same majority, let less emphatically, would agree with the normative role of logic, which, in the words of Feferman (see [32, p. 32), deals not with "how men actually reason but how they *strive* to do so" (italics mine).

To defend logic from accusations in psychologism, the logicians, starting from Charles Peirce and especially Gottlob Frege, have declared logic a normative discipline. This means that logic tells us how we *ought* to reason if we want to reason correctly. In the much talked-of book (see [52]) it is stated that logic is neither a normative nor a psychological theory. In other words, he has argued that actual reasoning, as "reasoned change in view", has nothing to do with logic. To the criticism of Harman's statements is devoted the paper [36], where it is explained why logic should be tied to norms of rationality. But the publication of an even more critical work (see [53] is already taken as a given if we consider where it is published.

Note the nice article [29][6], where the following questions are discussed:

(a) how do we reason?
(b) how ought we to reason?
(c) what justifies the way we ought to reason?

In the latter case the major role is given to the rules of introduction and elimination of logical constants as logical norms. Although in the context of very well founded concept of pluralism in logic, a serious problem arises. If logic is a normative discipline, then too many logical norms emerge. Engel's paper ends with the following notable words: "The gap between logic and the psychology of reasoning is not, on my view, as large as it is often claimed to be" (p. 234).

The return of psychologism to logic is one of the most significant tendencies of the development of modern logic. Surprisingly, to this question is devoted the Special Issue of one of the world's strictest logical journals (see [72]). Let us just reference the paper [10, p. 67], where Benthem talks about "understanding of 'psychologism' as a friend rather than an enemy of logical theory".

The return to psychologism is not accidental. Recently an exceptional development was obtained in *informal logic*, the movement that was born in North America in the 1970s. Informal logic is usually associated with everyday discourse, critical thinking, reasoning in ordinary language, studying of informal inference, and so on (see survey [48] and book [103]). Apparently, the case is that it has always been implicitly assumed that logic studies not all reasonings indiscriminately, but only the reasonings related to logic, i.e. it studies the logical reasonings. But in that case a pure tautology comes out: logic studies logic. In summary, it is the time to ultimately dismantle this tautology.

Interesting are also the tendencies that arise within mathematical logic itself. In the first place, it is the extraction of the necessary minimum of logical means, which leads to maximal generalization and abstraction of logic itself. In the work [18] a notion of 'abstract logic' was put into use, where an abstract logic is defined as a pair <*A*, *Cn*> such, that *A* is a universal algebra and *Cn* is a consequence (alias 'closure') operation on the carrier of A. A consequence operation *Cn* was introduced by Alfred Tarsky early in 1930.[7] In other terms, a Tarskian consequence relation is a binary relation (between sets of *L*-formulas and *L*-formulas), that satisfies the following conditions: *reflexivity*, *transitivity* and *monotonicity*. But nobody even tried to explain, why this topological closure operator's properties should determine some "kernel" of human reasoning.

Due to the fact that monotonicity property is counter-intuitive it has to be abandoned (or discarded at all), if we want to give a formal account of *defeasible reasoning* (see [70]). Concerning that it is difficult to find solid arguments against the properties of reflexivity and transitivity (although possible, if desired), the following definition of logic is not surprising (see [116, p. 136]): an abstract logic is defined as a pair <*A*, *Cn*> such, that a consequence operation satisfies only reflexivity and transitivity, in other words, "a logic is simply a *preorder*" (italics mine).

Definition of abstract logic suggested by Suszko has received further generalization and led to the notion of 'universal logic' (see [12], [13]). A universal logic is defined as a pair <*S*, |–> where *S* is some structure without any specification, and |– is a relation on *S*. Notice, that unlike *Cn* operation |– is not constrained, i.e. no axioms are stated for the consequence relation |–. Béziau's idea is that the relation of universal logic to all concrete logics is the same as of universal algebra to concrete algebras. Of course, the field of universal logic has arguably existed for many decades.

The term 'abstract logic' is also used in another sense, even contrary, maximally extending the notion of 'logic'. Such are the 'model-theoretic logics' (see [2]) which consist of a collection of mathematical structures, a collection of formal expressions of a language used to describe properties of such structures, and a relation of satisfaction between the two. The basic notion is that of satisfaction: $M \models \varphi$ if the expression φ is *true of*, or satisfied by, the structure *M*. The rigorous definition of abstract logic under the name 'general logics' is given in [27, pp. 27-28]. The structures can be very rich and so the construction of expressions, describing the properties of said structures, has much more expressive power than language of first-order logic. Hence, the problem of logical constants is not significant here. As stressed by Barwise: "We are primarily interested in logics where the class of structures are those where some important mathematical property is built in, and where the language gives us a convenient way of formalizing the mathematician's talk about the property" (see [3, pp. 4-5]). Note that the starting point for the study of abstract logics was Lindström's theorem (see above), and FOL itself is its simplest example. It is interesting that in the third edition of the famous book (see [23]) the new section is introduced under the title 'Lindström's Chatracterization of First-order Logic', which contains a definition of *abstract logic*, but more narrow than in [27].

As a whole, abstract logic associated with specific model-theoretic languages, aspires to overview the entire spectrum of logics. However, this tendency is also observed at the propositional level, where the main goal is set not as investigation of properties of a specific logic, no matter how interesting it is, but the whole classes of logics. The fourth chapter of the book ([22] is entitled 'From Logic to Classes of Logics', in it the classes of extensions of modal logics are regarded as lattices, and now the most important is the study of the properties of these lattices and various classes and subclasses of the elements of a given lattice, where the elements are logics themselves.

Currently the most impressive tendency of the development of modern logic is its intention towards unification of different logical systems and even whole movements. This phenomenon has received the name 'combining logics'. In the first book on this topic (see [19]) are presented general methods for combining logics, lots of examples and some suggested applications, including ones in Computer Science, where knowledge representation frequently requires the integration of several logical systems into a homogeneous environment. See also overview [20][8].

The latter of pointed out tendencies allows us to make the assumption that if logic has any relation to human thought process, then the level of human formal logicality lays hidden behind the 'functioning' of infinite classes of different logical systems. Or, in other words, we are on our way to combined reasoning. However, one thing may be absolutely positively stated: various discussions concerning the status and basic principles of logic, its current tendencies of development, tell us that logic stands in the face of grandiose changes and fundamental discoveries await us.

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Notes

1. It is English translation of the German version by J. H. Woodger. English translation of the Polish version by M. Stroińska and D. Hitchcock see in [121].
2. L. T. F. Gamut was a collective pseudonym for the Dutch logicians Johan van Benthem, Jeroen Groenendijk, Dick de Jongh, Martin Stokhof and Henk Verkuyl: “Any logical system which is appropriate as an instrument for the analysis of natural language needs a much richer structure than first-order predicate logic” (see [43, p. 75]).
3. The property of monotonicity states that if sentence A is a consequence of the set Γ then it is also a consequence of any set containing Γ as a subset (see [1]. Meaningfully, monotonicity indicates that learning a new piece of information cannot reduce the set of what is known. Classical first-order logic and many non-classical logics are monotony.
4. 2nd and 3rd volumes of ‘Handbook of Philosophical Logic’ (HPL) are nothing else but the overview of various non-classical logics: in the 2nd volume are considered the extension of classical propositional logic, for example, such as modal, temporal, deontic logic and others, and in the 3rd volume – the alternatives to classical logic, for example, such as multi-valued, intuitionistic, relevant logic and others. In the second edition of HPL (see [41]) this division is removed and a lot of other lines of non-classical logics is added. On completely new tendency in philosophical logic see [113].

5. See also [71] where the author considers five styles of deductive systems.
6. The paper was firs published in Italian in 2001.
7. See [118]. Here Tarski find the unexpected use for the closure operator to study abstract consequence relation. This work is preceded by his papers on logical consequence (see above), which is not always acknowledged.
8. However, let us note that first examples of combined logical systems appeared in middle the 1950s, when Rasiowa in [97] has obtained a product of two-element matrix for classical equivalence and three-valued matrix for Łukasiewicz's implication and has given the axiomatization for the resulting new six-valued matrix. In turn, Prior in [95] gives the first examples of combined modal-temporal logics.

Studia Humana
Volume 3:1 (2014), pp. 27—37
DOI: 10.2478/sh-2014-0003

Ordinal Or Cardinal Utility: A Note

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Abstract:
Modern microeconomic theory is based on a foundation of ordinal preference relations. Good textbooks stress that cardinal utility functions are artificial constructions of convenience, and that economics does not attribute any meaning to "utils." However, we argue that despite this official position, in practice mainstream economists rely on techniques that assume the validity of cardinal utility. Doing so has turned mainstream economic theorizing into an exercise of reductionism of objects down to the preferences of 'ideal type' subjects.

Keywords: Ordinal, cardinal, utility

We thank William Barnett II and Robert P. Murphy for suggestions which greatly improved our paper. All remaining errors and infelicities are of course our own responsibility.

1. Introduction

This paper will attempt to demonstrate that the ordinal utility of Austrian economics is the only version of this concept compatible with human action [42]. It is our contention that although mainstream or neoclassical economists do utilize ordinal utility, they do so only superficially. More basically, they cleave to cardinal utility, which, we maintain, is incompatible with a proper analysis of utility. In section II we discuss these two very different understandings of utility. Section III is devoted to an analysis of a case in point, wealth redistribution. The purpose of section IV is to deal with indifference curves: we utilize them to defend ordinal utility and disparage the cardinal variety. The focus is on subjectivism in section V. We conclude in section VI.

2. Ordinal and Cardinal Utility

Ordinal utility may be defined as ranking, and/or setting aside in human action, or choice. Etymologically, ordinality in this context stems from ordering, preferring. For example, here might be the rank ordering of Mrs. Smith, consumer: 1st: 10 eggs; 2nd: $5; 3rd: 9 eggs; 4th: $4; 5th: 8 eggs; 6th: $3...[1]

There is no dispute about this amongst economists. In Hicks' [31] seminal discussion of Pareto, the old utilitarian notions were thrown overboard. No longer did mathematical economists need to rely on dubious assumptions that troubled philosophers; the Law of Demand and other components of consumer theory could be reformulated with a purely ordinal foundation. There are no extant cases where a member of the dismal science in good standing rejected this concept, or denied that we are capable of such orderings [54, p. 6]. Ordinal utility, then, is one of the pillars of the modern dismal science.

Cardinal utility, in very sharp contrast, is a different matter indeed. Here, numerical measures of utility are assigned to different goods, services, objects. For example, one might say that for Mr. Jones, a pencil offers him 5 utils (units of happiness or utility), a wrist watch 10 utils, and a shirt, 20 utils. Since these are all cardinal, or objective numbers, it is thus possible to perform mathematical operations on them. For instance, based on these cardinal numbers, we are entitled to infer that for Jones, a shirt is equivalent, in terms of utility, to two watches, or to four pencils; that two pencils are worth, to him, one watch.[2]

Very few economists accept cardinal utility, at least the rather simplistic or elemental version of it we have so far discussed. They full well realize that, while there are indeed objective measures of length (inches, meters, miles), weight (pounds, kilograms), speed (miles per hour), etc., there are no such objective measures of happiness or utility, such as utils. These are merely a heuristic device, so to speak, for most professionals in the discipline.[3]

3. Wealth Redistribution

There are of course exceptions. For example, one of the justifications for income redistribution from rich to poor can be seen in diagram #1. Here, utility appears explicitly in the form of "utils," on the y axis, while money, or wealth, is depicted on the x axis. The downward sloping curve illustrates decreasing marginal utility, from which we can deduce that a dollar taken away from the rich person, B, and given to the poor person, A, will increase total utility, in that the last dollar spent by B yields him less satisfaction than the marginal dollar spent by A.

According to Pigou [46, p. 89]: "…it is evident that any transference of wealth from a relatively rich man to a relatively poor man of similar temperament, since it enables more intense wants to be satisfied at the expense of less intense wants, must increase the aggregate sum of satisfaction."[4] If this is not the employment of cardinal utility in the context of technical economics, it is difficult to know what would be.

This treatment is highly problematic in that not only does it embrace explicit cardinal utility, it also engages in interpersonal comparisons of utility. It is one thing to claim that Jones values his shirt twice as highly as his watch; it is quite another, and even more fallacious if possible, to take the position that Jones derives half as much satisfaction from his pencil as does Smith from her egg.

Nor is this wealth redistribution argument the only case on record where cardinal utility is embraced so explicitly.[5] But these are few and far between, and thus less harmful to our profession; they are not endemic.

4. Indifference Curves

The same cannot be said, unfortunately, for indifference curve analysis. This practice is so ubiquitous that no citation is even offered. How, then, do indifference curves logically imply cardinal utility?

At first glance, this is not the case. Far from it. In diagram #2 we depict the "usual" indifference curve set. Note that the three indifference curves are labeled 5, 6, and 7 utils, respectively, in increasing order as we move away from the origin. The argument for the interpretation of this graph in terms of ordinal, not cardinal utility is the following: the numbers, 5, 6 and 7 serve, merely, as markers, in this context. And, as such, they are clearly ordinal not cardinal. For example, we could have labeled the three indifference curves, instead, 50, 60 and 70 utils, or 500, 600 and 700 utils, or, for that matter, 1, 2 and 3 utils, or 10, 20 and 30 utils, and it would not have made any nonevermind. All that any of the labels would have indicated is that the indifference curve closest to the origin yields the least utility, the one furthest away, the most, and the one in the middle takes on an intermediate role in this regard. "If that isn't ordinal utility, what is?" might argue the advocate of interpreting indifference curves solely as ordinal.

This argument however, moves too fast. This can be seen by focusing on point *C*, where the budget line and indifference curve "6" are tangent to one another. The algebraic interpretation of this joinder is of course:

$$(1) \qquad \frac{MUy}{MUx} = \frac{Py}{Px}$$

There is no problem with the right side of this equation. The prices of *Y* and *X*, respectively, are properly cardinal numbers, and the usual mathematical operations (division in this case) may be performed on them. Matters are far more difficult with the left side of this equation. For, here, we are *dividing* one number by another number, and, this can only be done with regard to cardinal numbers, not ordinal ones. For example, it is mathematically correct[6] to divide the cardinal number 100 by the cardinal number 50 and arrive at the cardinal number 2. But, what are we to say of an attempt to divide the *ordinal* number 100th, by the *ordinal* number 50th, and derive the *ordinal* number 2nd? This would be an utter impossibility. Indeed, it would be mathematical gibberish.

Nor will there be any improvement in such matters merely by transposing equation (1) into (2):

$$(2) \qquad \frac{MUy}{Py} = \frac{MUx}{Px}$$

If anything, there is now a worsening. For in equation (1), at least the right side of it achieves mathematical legitimacy. Not so in equation (2). For, it is illicit to divide an ordinal number by a cardinal one. For example, the mathematical phrase, "18th divided by 3" succeeds in yielding only a literally meaningless statement. It is certainly *not* true that "18th divided by 3" is equal to the cardinal number 6, nor, yet, to the ordinal number 6th. On the contrary, it is quite literally meaningless.[7]

Thus, we can see that the mere *labeling* of the indifference curves masks the underlying reality. Yes, the nomenclature utilized in marking them appears, superficially compatible with ordinal utility. After all, if 5, 6 and 7 serve, merely, as markers, and could be substituted for by 50, 60 and 70 utils, or 500, 600 and 700 utils, or, 1, 2 and 3 utils, or 10, 20 and 30 utils, then this is all compatible with ordinality. However, this is not so; indeed, cannot be correct. For, given what the tangency point tells us, there are and must of necessity be cardinal numbers involved in this technique. How else could the mathematical operations performed on them in equations 1 and 2 be coherent?

Neo classical economics is thus challenged with a dilemma: either eschew ordinal utility, or jettison indifference curves.[8] Cardinal utility and indifference curves go together; you can't have one without the other. Instead, the challenge taken up by mainstream economists is to find some way of squaring this particular circle with ever more sophisticated quantitative techniques at higher levels of abstraction.

The history of the nature of utility has been a checkered one of moving backwards and forwards between cardinalism and ordinalism. Of the three founders of the law of declining marginal utility, Jevons (1871) was probably the most explicit in defining utility in hedonistic terms that could later be made amenable to quantitative techniques of differentiation. Walras (1874), whose project was to derive a pure quantitative approach to the theory of value, deferred to a numeraire from which cardinal utility could be inferred. Walras essentially turned the problem into an objective function by asking the question: based on a given state of endowments, what should the exchange values be in order to ensure the continuation of current production by avoiding any income distribution effects – his "theorem of equivalent redistributions"? Only Menger (1871) remained true to an ordinal conception of the problem, arguing that quantitative techniques alone could never solve the problem posed by the interposition of subjective individuals amongst their objects of choice. Walras essentially cut out the subjective individual from his equations.

The supposed acceptance of ordinalism by modern neoclassical economists could never quite rid itself of the implicit cardinal use made of utility functions or attempts by Lange (1934) and von Neumann and Morgenstern (1944) to cardinalise ordinal utility for interpersonal utility comparisons. Cardinalisation is essential for the application of quantitative methods.

At higher levels of abstraction it is argued that the indexes of ordinal utility can be cardinalised. Lange (1934) tried to do this first by obtaining preferences not only of consumer bundles, but also of the movements between bundles. Lange initiated a series of discussions on the determinateness of the utility function. He tried to prove that from two postulates the measurability of utility is guaranteed: (1) given any two combinations of consumer's goods, the consumer is able to state that one is preferred to the other or equally preferred and (2) given any four combinations of consumers' goods, the individual is always able to place the movements in ordinal relationship. Lange's weakness was his assumption of linear transformation involving scale and origin constants. However, he was later to be proved wrong by Samuelson (1938).[9]

Later the Neumann-Morgenstern cardinal utility for interpersonal utility comparisons used in game theory was derived by the application of probability theory and the accounting of risk preferences. Willingness to pay for lottery tickets with different probabilities of different bundles containing an individual's preferences are used to derive a cardinal measure of utility. Arrow (1950) finally demonstrated all such measures as problematic for welfare economics and the field has been in disarray ever since [49].

The mainstream of the economics profession plays lips service to the fact that utility is ordinal, but by means of indexing implicitly adopts cardinal utility in its application to theory. For example, indices of utility are derived from prices on the basis that MU1/MU2 = P1/P2. It is then maintained that a higher derived utility is merely expressive of a higher ranking, as opposed to adopting 1st, 2nd, etc for the marginal utilities (MUs). But if this were true, then MU1/MU2 = 20/15 is 'ordinally' equivalent to MU1/MU2 = 18/15; but both ratios cannot be equated with a single cardinal ratio for P1/P2.

However one tries to resolve the matter, one is always left with the impression of trying to square the circle between utility and price. After all, isn't utility the basis of price? Therefore must there not be a way from the one to the other that proves reconcilable? The positing of such questions is logical enough, but they are not rhetorical! They are not rhetorical because the two sides of the equation are incommensurate with one another. – This holds true not only logically because of the impossibility of performing mathematical operations upon ordinal rankings relative to cardinal measures as argued above. It also obtains more fundamentally because, whereas the right-hand side of the equation refers to objects (of price), the left-hand side refers to an individual subject (who is doing the preference ranking). How else can such a ranking be constituted? Ordinality implies judgment and judgment requires a subject to make such judgments. In the market context, there is no representative individual doing a ranking on the basis of some consensus standard of ordering.[10] Thus judgment and ordinality is inherently a subjective phenomenon.[11]

One possible approach to reconciling the two sides is by viewing the left hand side as related to goods as on the right hand side. But in that case, if we are to eschew all notions of 'intrinsic' value of objects as independent of their evaluators, viz, their subjective users, we are back at cardinal utility with the need for some form of interpersonal utility comparison in order to perform the necessary aggregations for the purposes of deriving their marginal value – at the margin of the aggregate for the particular goods in question. Such an approach turns the reconciliation into a superficial one (or one of convenience for the application of quantitative methods) that eschews subjectivism by embracing the 'intrinsic' value concept as the basis for quantitative theories through the back door. The variety and uniqueness of individual preferences are suppressed by the representative agent and invariant preference constructs are underpinning mainstream models. In reality, the values of individual goods *are not* independent of the act of valuation of their evaluators.[12] Prices do not represent any form of measure pertaining to the goods in question, but are rather expressions of valuation that convey useful information about current and planned (anticipated) arrangements of goods for economic action. Seen in this light, neoclassical indifference curve analysis is a pure (and unwarranted) reductionism of price to utility and from object to (an ideal) subject.

5. Subjectivism

Attempts at quantification have led to the analysis being increasingly focussed upon the objects, withdrawing attention ever further away from the subjective element that constitutes the basis of price formation. The cost of such withdrawal is the necessity of restrictive assumptions that turn against reality (such as transitivity[13] and invariance with respect to time, which are all bound up with the problem of judgment and choice in the first place), thereby turning price theory increasingly into an empirical "science" devoid of subjective content.

Such content, however, becomes vital to the ability of prices to transmit information in the market context; to realise that prices are expressions of individual wants in relation to availability. To ignore the subjective basis of price theory is to fail to appreciate the informational surrogate role of prices. It not only provides knowledge about the relative scarcity of goods in meeting individual wants (utilities), but also information that affects individual preferences. It does so in a way that can never be regarded as "given" in any of the senses that it has conventionally become necessary to "fix" before being able to apply indifference curve analysis to it.

So whereas the quantitative theorist can agree that utility is ordinal, he simply evades the problem of aggregation, having attempted to transform its necessarily subjective basis into an objective one. His indifference curves are then based on empirical data sets, which render them sterile or of only historical interest. If the neoclassical economist intends to apply this technique generally – which he most certainly does– he can only do so predicated on arbitrary restrictive assumptions. He goes too far when, knowing that marginal utility must form the basis of price formation, he forgets all about his restrictive assumptions and tries to apply his indifference curves to marginal utility considerations[14], for example, when trying to justify the higher value of a dollar to a poor person relative to a rich person.

6. Conclusion

No reconciliation is possible between Austrian and mainstream economists on this matter because of the limitations of the quantitative method which can only deal with the objective phenomena surrounding the subjects that dispose over them (through ownership and control). The quantitative neoclassical theorist necessarily eschews a subjectivist approach to the problem. He nonetheless hangs onto the only notion of subjectivity which he believes can be integrated into a pure quantitative theory or provides it with an interface: the notion of indifference. For if one can be indifferent between two things, does that not necessarily imply a measure of equality? But given the nature of ordinal choice and its relation to the subject (not merely a relation of measure between objects), this theory can say very little about the objects that fall on either side of the indifference map of individual choice. This is because of the arbitrary restrictive assumptions upon which such reasoning (explicitly or implicitly) must rest. The measure of equality is accidental (in the sense that it is place and time bound). It lacks general applicability because of the arbitrary assumption of having to maintain a static state of welfare that can be traced back to Walras' theorem of equivalent redistributions. Such a condition does indeed provide an objective solution to the problem; if only it could be made to stick in the real world.

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Notes

1. Such rank orderings were utilized by Menger (1950), Rothbard (1993). For the claim that they are incompatible with the Misesian (1998) notion of singularism, see Barnett and Block (2008).
2. Note that indifference is implied by cardinal, but not ordinal utility. Or, at the least, cardinal utility is logically compatible with indifference (two goods, or combinations thereof, as on an indifference curve, yield an equal amount of cardinal utility, and we are thus indifferent between the two of them). In the latter case, there is only preference, not indifference; in the former, Jones is indifferent between one shirt, or two watches, or four pencils. For a defense of indifference, see Caplan, 1999, 2000, 2001, 2003, 2008. For a critique: Block, 1999, 2003, 2005, 2007; Callahan, 2003; Carilli and Dempster, 2003; Hoppe, 2005, 2007; Hulsmann, 1999, Machaj, 2007; Murphy, 2008; Stringham, 2001, 2008; Stringham and White, 2004. More generally, see Murphy, Wutscher and Block, 2010.
3. This applies, even, to those who specialize in "happiness studies." See for example *Journal of Happiness Studies*. Nowhere in this literature can be found a claim to the effect that happiness itself, or utility per se rose or fell by 2.3% or by any other such cardinal number. Rather, utility is operationally defined as an amalgamation of answers to questions on the part of specific people at certain times and places, and, as the numbers that result from these surveys are indeed cardinal, it is entirely legitimate to say that satisfaction rose or fell by a certain percentage between any two given surveys, either at different times or places or both.
4. Cited in Gordon, 1993.
5. Utils also appear on the vertical axis on numerous occasions in the economics literature. These are clearly "smoking gun" instances of the fallacious employment of cardinal, not ordinal utility in the mainstream economics literature. See on this Barnett (2003) who mentions several such cases.
6. This phrase is somewhat unsettling, given practices in academia and elsewhere with regard to "correctness."
7. There is of course a sense in which a number ending in a "th" can be and indeed is a legitimate cardinal number, not an ordinal one. For instance, the number 1/18, and pronounced "one eighteenth" is a perfectly acceptable number in mathematics. But it is cardinal, not ordinal.
8. We have no objection to the concept of "indifference" itself. This word is a perfectly acceptable one in the English language. Everyone knows precisely what it means. The present authors, too, are accustomed to employing it. Our objection arises with its use as a matter of economic science. An analogy may make this clear. In physics, "work" equals force time distance. But if even a top athlete holds bar bells of pretty much any weight, even as little as five pounds at arm's distance, he will soon tire. Will he be doing any "work?" Not in the technical sense of physics. But in ordinary language, as we see the sweat on his brow from this exercise, all would agree with the claim that he is working very hard indeed. It is the same with "indifference." Unobjectionable in ordinary language, but not in the technical language of economics.

9. Lange's theorem rested on the critical assumption that agents' ordinal utility functions are linear under the transformation to preferences over readjustments. Samuelson showed that there is no a priori reason to assume why an individual's preference scale should obey such an arbitrary restriction.
10. Not only do individuals rank things differently and in incompatible ways, but some do not rank at all (at least not in the rational sense) according to some. For example Jung (1971) classifies individuals into four broad functional categories, arguing that for any individual to function coherently in the world he has to develop one of them as his superior function to which the other functions become subordinate (separately as inferior and auxiliary functions) whenever a conflict in the rankings in the context of human action arises. Thus there are the rational types, who are either differentiated thinking or feeling individuals whose value rankings are predicated upon one or the other of these two functions, generally suppressing the other whenever a conflict arises. In contrast; there are the non-rational, but perceptive types, the sensation and intuitive individuals, whose value systems are not based on rankings *per se*, but upon the intensity of their experiences, seeming wholly irrational to the rational types (but that may nonetheless use one of the rational functions as an auxiliary function in order to communicate coherently with others). This analysis is in sharp contrast to the Austrian view of rationality as purposefulness (see on this Mises, 1998 and Kirzner, 1973.)
11. States Hayek (1979, 52): "And it is probably no exaggeration to say that every important advance in economic theory during the last hundred years was a further step in the consistent application of subjectivism." Also, see the following on this issue: Barnett, 1989; Block, 1988; Buchanan and Thirlby, 1981; Buchanan, 1969, 1979; Butos and Koppl, 1997; Cordato, 1989; DiLorenzo, 1990; Garrison, 1985; Gunning, 1990; Kirzner, 1986; Mises, 1998; Rizzo, 1979, 1980; Rothbard, 1979, 1997; Stringham, 2008.
12. Without using it as analogy, this conclusion has an uncanny resonance with the implications of wave-particle theory of quantum physics, where the act of observation is seen to influence the results of experiments.
13. For a critique of this concept from an Austrian point of view, see Block and Barnett, 2012
14. As opposed to simple – but admittedly more sterile – marginal rates of substitution analyses

Diagram 1

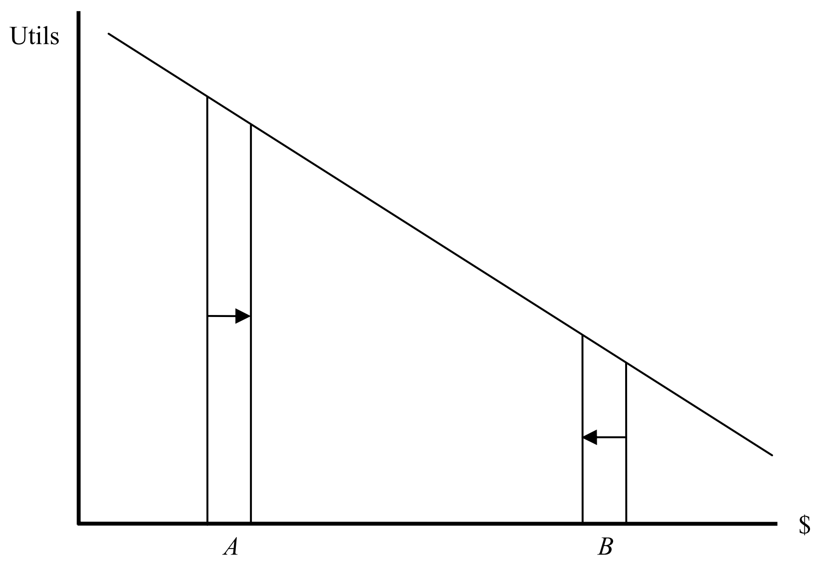
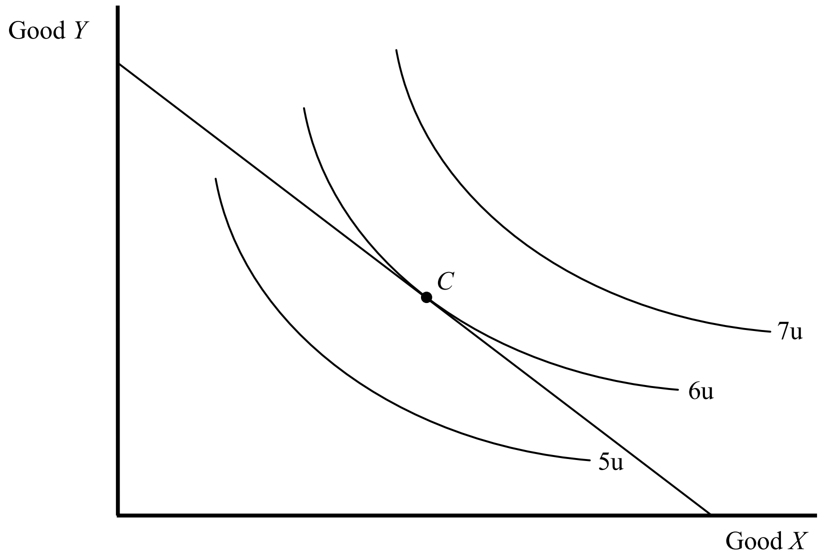


Diagram 2



Studia Humana
Volume 3:1 (2014), pp. 38—47
DOI: 10.2478/sh-2014-0004

Continuous Logic and Scheduling in Systems with Indeterminate Processing Times

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Abstract:
A general approach to the synthesis of an optimal order of executing jobs in engineering systems with indeterminate (interval) times of job processing is presented. As a mathematical model of the system, a two-stage pipeline is taken whose first and second stages are, respectively, the input of data and its processing, and the corresponding mathematical apparatus is continuous logic and logic determinants.

Keywords: system, optimal order of jobs, continuous logic

1. Introduction

The study of engineering systems begins with determining the dependence of the performance factor of system on its parameters. This dependence can be used for estimating the performance factor of the system, to analyze it qualitatively, to optimally synthesize the system, etc. As a rule, the existing estimation methods for performance of engineering systems are oriented only to calculating performance factor of system and are not meant for their analysis and synthesis [1].

In [2, 3] there is an approach for studying various systems based on pipeline model of scheduling theory and on the mathematical apparatus of continuous logic and logical determinants, which makes it possible to derive an observable and easily calculated expression for system performance and to carry out qualitative analysis of the effect of system parameters on its performance and on its optimal synthesis according to the criterion of best performance. In this case the time parameters are assumed to be deterministic. In practice, these parameters are in many cases nondeterministic, which substantially hampers the study of system.

We consider an extension of the general approach for optimal synthesis of engineering systems with uncertain (interval) type time parameters to nondeterministic case [4]. Under application of this approach to optimal synthesis of engineering systems with interval time parameters this problem reduces to solving similar problems for two systems with deterministic time parameters equal to the upper and lower bounds of the corresponding intervals.

2. Problem Statement

Consider a system operating in batch mode and let the batch contain n different jobs $1,...,n$. We employ simplest two-phase model of system. So, in first phase performed by first system unit first operation, namely inpuiting of the initial data, is carried out; further, in other phase which performed by the second unit of system the second operation is carried out – transformation and processing of these data in various functional units of the system (processor, main memory, and external memory) and the output the result. The units are assumed to operate consecutively. Each job i $(i = \overline{1,n})$ firstly goes to the first unit, where first operation is full performed, and after that goes to second unit, where the second operation is carried out completely.

The time of execution of the first operation on arbitrary job i is assumed to be known inexactly and to be determined by a closed interval $\widetilde{a}_i = [a_{i1}, a_{i2}]$ of all possible values of this time. In similar way the time of execution of the second operation on job i is set: $\widetilde{b}_i = [b_{i1}, b_{i2}]$. So, the first unit starts the execution of the current job immediately after end of the previous job, i.e., it operates without idle times, whereas the second unit starts the execution of the current job j only after the job j leaves the first unit, i.e., in the general case it operates with idle times. It is required to choose an order of jobs in the system under which its best performance is ensured, i.e., ttotal execution time of all jobs is minimum.

As in deterministic case [5, 6], the optimal order of jobs can be assumed to be permutable, i.e., jobs must pass through two units in same order. Assume that execution times of first and second operations on an arbitrary job i are exact and are equal to a_i and b_i, respectively. Let for a pair of jobs (i, j) the order of passage through the first unit be $i \to j$, and the order of passage through second unit be opposite: $j \to i$. Let us change the order of jobs passing in the first unit by placing job i after j and moving job j (together with the jobs located earlier between i and j) to the left by length of freed time interval a_i. In this case the interval of the execution of one of the jobs i, which are subject to permutation, is moved to the right. However, it then ends at the time of completion of the execution of the job j in the first unit (before permutation, i.e., as previously, before the time of beginning of the execution of job in the second unit). Hence, a change in the order of jobs in the first unit does not affect the sequence of jobs in the second unit. Therefore, the same order of passage of jobs through the two units can be chosen without changing the resultant time of execution of all jobs. It means that for deterministic execution times of operations the optimal order in the sequence of jobs passing can be sought within the set of permutational orders of jobs. This conclusion is true for arbitrary deterministic execution times a_i and b_i of operations inside given intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$. Consequently, in accordance with the conditions of the problem, it remains valid if times of operations are assumed to be equal to the indicated interval values.

Thus, the solution of the stated problem reduces to finding an external permutation

$$P_n = (i_1, i_2, .., i_n), \quad i_k \in \{1,2,..., n\}, \tag{1}$$

of n given jobs that determines the order of jobs in the system, which is the same for its two units. The symbol i_k in expression (1) is the index of the job occupying the k-th place in the ordered sequence.

3. Logic Algebra of Nondeterministic Quantities and their Comparison

The problem solution requires some facts of the logic of nondeterministic interval quantities and of comparison theory for these quantities [4]. We shall proceed from continuous logic for deterministic (point) quantities [7]. The basic logical operations on these points are disjunction $\vee$ and conjunction $\wedge$ that are defined in following formulas:

$$\begin{aligned} a \vee b &= \max(a,b), \\ a \wedge b &= \min(a,b) \end{aligned} . \tag{2}$$

Here $a, b \in C$, and the set C is an arbitrary interval of real numbers. Operations (2) obey the majority of laws of discrete logic, namely

$$a \vee a = a, \quad a \wedge a = a \qquad \text{(tautology)} \tag{3}$$
$$a \vee b = b \vee a, \quad a \wedge b = b \wedge a \qquad \text{(commutative law)} \tag{4}$$
$$(a \vee b) \vee c = a \vee (b \vee c), \quad (a \wedge b) \wedge c = a \wedge (b \wedge c) \qquad \text{(associative law)} \tag{5}$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad \text{(distributive law)} \tag{6}$$
$$a \vee (a \wedge b) = a, \ a \wedge (a \vee b) = a \tag{7}$$
$$a + (b \genfrac{}{}{0pt}{}{\vee}{\wedge} c) = (a + b) \genfrac{}{}{0pt}{}{\vee}{\wedge} (a + c), \tag{8}$$
$$a - (b \genfrac{}{}{0pt}{}{\vee}{\wedge} c) = (a - b) \genfrac{}{}{0pt}{}{\wedge}{\vee} (a - c), \tag{9}$$
$$a \cdot (b \genfrac{}{}{0pt}{}{\vee}{\wedge} c) = (a \cdot b) \genfrac{}{}{0pt}{}{\vee}{\wedge} (a \cdot c), \quad a, b, c > 0, \tag{10}$$
$$-a \cdot (b \genfrac{}{}{0pt}{}{\vee}{\wedge} c) = (-a \cdot b) \genfrac{}{}{0pt}{}{\wedge}{\vee} (-a \cdot c), \quad a, b, c > 0, \tag{11}$$

A special partial case of the equation (11) for $a = 1$ is the following law:

$$-(b \genfrac{}{}{0pt}{}{\vee}{\wedge} c) = (-b) \genfrac{}{}{0pt}{}{\wedge}{\vee} (-c), \tag{12}$$

We now pass to continuous logic for interval quantities. In this case the continuous-logical operations of disjunction and conjunction (2) are generalized as set-theoretic constructions:

$$\begin{aligned} \tilde{a} \vee \tilde{b} &= \{a \vee b \mid a \in \tilde{a}, b \in \tilde{b}\}; \\ \tilde{a} \wedge \tilde{b} &= \{a \wedge b \mid a \in \tilde{a}, b \in \tilde{b}\}. \end{aligned} \tag{13}$$

Here $\tilde{a} = [a_1, a_2]$ and $\tilde{b} = [b_1, b_2]$ are intervals regarded as the corresponding sets of points (values) belonging to them. According to [4], operations on intervals (13) obey the same laws (3)–(12) as the operations on point quantities (2). In particular, distributive laws (8) and the law (12) take form:

$$\tilde{a} + (\tilde{b} \genfrac{}{}{0pt}{}{\vee}{\wedge} \tilde{c}) = (\tilde{a} + \tilde{b}) \genfrac{}{}{0pt}{}{\vee}{\wedge} (\tilde{a} + \tilde{c}), \tag{14}$$
$$-(\tilde{b} \genfrac{}{}{0pt}{}{\vee}{\wedge} \tilde{c}) = (-\tilde{b}) \genfrac{}{}{0pt}{}{\wedge}{\vee} (-\tilde{c}). \tag{15}$$

Due to [4] the results of the logical operations of disjunction and conjunction on intervals (13) are calculated by the formulas

$$\tilde{a} \vee \tilde{b} = [a_1, a_2] \vee [b_1, b_2] = [a_1 \vee b_1, a_2 \vee b_2], \tag{16}$$
$$\tilde{a} \wedge \tilde{b} = [a_1, a_2] \wedge [b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2]. \tag{17}$$

We briefly present some important facts of comparison theory for intervals. [4]

1. For any pair of intervals $\tilde{a} = [a_1, a_2]$ and $\tilde{b} = [b_1, b_2]$ the equivalence relation

$$(\tilde{a} \vee \tilde{b} = \tilde{a}) \Leftrightarrow (\tilde{a} \wedge \tilde{b} = \tilde{b}), \tag{18}$$

holds, i.e., like point quantities, the intervals are compatible (in the sense that if one of the two quantities is maximal, then the other is minimal and vice versa).

2. Pairs of intervals $\tilde{a} = [a_1, a_2]$ and $\tilde{b} = [b_1, b_2]$ can be in relations «greater than» and «smaller than» defined in the same way as in the case of point quantities by the such equivalence:

$$(\tilde{a} \geq \tilde{b}) \Leftrightarrow (\tilde{a} \vee \tilde{b} = \tilde{a}, \tilde{a} \wedge \tilde{b} = \tilde{b}). \tag{19}$$

3. In accordance with (19), any two intervals $\tilde{a}$ and $\tilde{b}$ that are in relation $\tilde{a} \geq \tilde{b}$ or $\tilde{a} \leq \tilde{b}$ are said to be comparable. Otherwise $\tilde{a}$ and $\tilde{b}$ are incomparable.

4. For intervals $\tilde{a} = [a_1, a_2]$ and $\tilde{b} = [b_1, b_2]$ to be comparable and satisfy the relation $\tilde{a} \geq \tilde{b}$ it is necessary and sufficient that system of inequalities $(a_1 \geq b_1, a_2 \geq b_2)$ holds, and for $\tilde{a}$ and $\tilde{b}$ to be incomparable it is necessary and sufficient that at least one of systems of inequalities $(a_1 < b_1, a_2 > b_2)$ or $(b_1 < a_1, b_2 > a_2)$ are true. Thus, only the intervals displaced relative to each other along number axis are comparable; in this case interval displaced to the right is greater. If one of intervals overlaps other the intervals are incomparable.

5. In a system of intervals $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_k$ the interval $\tilde{a}_1$ is said to be maximal (minimal) interval if it is comparable with other intervals $a_2, ..., a_k$ and is in relations $\tilde{a}_1 \geq \tilde{a}_2, ..., \tilde{a}_1 \geq \tilde{a}_k$ $(\tilde{a}_1 \leq \tilde{a}_2, ... \tilde{a}_1 \leq \tilde{a}_k)$ with them.

6. It is necessary and sufficient in system of intervals $\tilde{a}_1 = [a_{11}, a_{12}], \tilde{a}_2 = [a_{21}, a_{22}], ..., \tilde{a}_k = [a_{k1}, a_{k2}]$ for interval $\tilde{a}_1$ be maximal that the system of the relations holds:

$$a_{11} = \bigvee_{i=1}^{k} a_{i1}, \quad a_{12} = \bigvee_{i=1}^{k} a_{i2}, \tag{20}$$

and for $\tilde{a}_1$ to be minimal it is necessary and sufficient that following equations is true:

$$a_{11} = \bigwedge_{i=1}^{k} a_{i1}, \quad a_{12} = \bigwedge_{i=1}^{k} a_{i2}, \tag{21}$$

4. Derivation of Optimality Conditions

In the previous case we define a relationship between the execution times $\tilde{a}_i, \tilde{b}_i, \tilde{a}_j, \tilde{b}_j$ of two arbitrary jobs (i, j) under which they must be executed in order $i \to j$ in optimal sequence of jobs $P(n)$ (1). Let $P_k = (i_1, ..., i_k); k \leq n$, be initial section of P_n and let $\tilde{t}_1(P_k)$ and $\tilde{t}_2(P_k)$ be time intervals containing all possible times of completion of sequence P_k in 1st and 2nd units. Because $P_{k+1} = (P_k, i_{k+1})$, we can write

$$\tilde{t}(P_{k+1}) = \tilde{t}_1(P_k) + \tilde{a}_{i_{k+1}}, \quad \tilde{t}_2(P_{k+1}) = [\tilde{t}_1(P_{k+1}) \vee \tilde{t}(P_k)] + \tilde{b}_{i_{k+1}}. \tag{22}$$

Here $\vee$ is disjunction of type (13). The recurrence relations (22) make it possible to calculate the total time of execution for any order of the sequence of jobs P_n in form of a time interval $\tilde{T}(2, n) = \tilde{t}_2(P_n)$. Let $P_n^1 = (i_1, ..., i_k, i, j_1, ..., j_n)$ and $P_n^2 = (i_1, ..., i_k, j, j_1, ..., j_n)$ be two sequences of jobs passing through the system that differ only in order of execution of jobs i and j occupying the $(k+1)$ -th and $(k+2)$ -th positions in sequence. Let us find out when P_n^1 is more preferable than P_n^2, i.e.

when jobs i and j must be executed in order $i \to j$ (and not vice versa). The corresponding condition is written as

$$\widetilde{t}_2(P^1_{k+2}) \le \widetilde{t}_2(P^2_{k+2}) . \tag{23}$$

According to (22), the sequence P^1_n is more preferable than P^2_n if the time or passage of its regulated subsequence P^1_{k+2} through two units is less than that of P^2_{k+2}. To write preference condition in explicit form we must express $\widetilde{t}_2(P_{k+2})$ via the time parameters $\widetilde{a}_i$ and $\widetilde{b}_i$ of jobs. Let $\widetilde{t}_1(P_k) = \widetilde{t}$. Then $\widetilde{t}_2(P_k) = \widetilde{t} + \widetilde{\Delta}$, where $\widetilde{\Delta} = \widetilde{b}_{i_k}$. By the fact that $P^1_{k+1} = (P_k, i)$ and $P^1_{k+2} = (P^1_{k+1}, j) = (P_k, i, j)$, on applying twice recurrence relations (22) we obtain

$$\widetilde{t}_1(P^1_{k+1}) = \widetilde{t} + \widetilde{a}_i;\ \widetilde{t}_2(P^1_{k+1}) = ((\widetilde{t} + \widetilde{a}_i) \vee (\widetilde{t} + \widetilde{\Delta})) + \widetilde{b}_i;$$

$$\widetilde{t}_1(P^1_{k+2}) = \widetilde{t} + \widetilde{a}_i + \widetilde{a}_j;$$

$$\widetilde{t}_2(P^1_{k+2}) = \{(\widetilde{t} + \widetilde{a}_i + \widetilde{a}_j) \vee [((\widetilde{t} + \widetilde{a}_i) \vee (\widetilde{t} + \widetilde{\Delta})) + \widetilde{b}_i]\}.$$

We similarly determine haracteristics P^2_{k+1} and P^2_{k+2}; in this case we have

$$\widetilde{t}_2(P^2_{k+2}) = \{(\widetilde{t} + \widetilde{a}_i + \widetilde{a}_j) \vee [((\widetilde{t} + \widetilde{a}_i) \vee (\widetilde{t} + \widetilde{\Delta})) + \widetilde{b}_j]\} + \widetilde{b}_i .$$

The substitution of the above expressions into the formula (23) yields explicit form of the condition under which the jobs i and j in the optimal sequence must follow in the order $i \to j$:

$$\{(\widetilde{t} + \widetilde{a}_i + \widetilde{a}_j) \vee [((\widetilde{t} + \widetilde{a}_i) \vee (\widetilde{t} + \widetilde{\Delta})) + \widetilde{b}_i]\} + \widetilde{b}_i \le \{(\widetilde{t} + \widetilde{a}_i + \widetilde{a}_j) \vee [((\widetilde{t} + \widetilde{a}_j) \vee (\widetilde{t} + \widetilde{\Delta})) + \widetilde{b}_j]\} + \widetilde{b}_i . \tag{24}$$

To simplify inequalities (24) we apply the laws (8), (12) and we can take by (8) the term $\widetilde{t}$ outside the parentheses on both sides of (24). On canceling it, we find

$$\{(\widetilde{a}_i + \widetilde{a}_j) \vee [(\widetilde{a}_i \vee \widetilde{\Delta}) + \widetilde{b}_i]\} + \widetilde{b}_j \le \{(\widetilde{a}_i + \widetilde{a}_j) \vee [(\widetilde{a}_i \vee \widetilde{\Delta}) + b_j]\} + \widetilde{b}_i .$$

We now take the terms $\widetilde{a}_i, \widetilde{a}_j, \widetilde{b}_i$ and $\widetilde{a}_i, \widetilde{a}_j, \widetilde{b}_j$ outside the curly brackets on left- and right-hand sides of the new inequality, respectively. On canceling the common terms on the two sides we write

$$(-\widetilde{b}_i) \vee (\widetilde{\Delta} - \widetilde{a}_i - \widetilde{a}_j) \vee (-\widetilde{a}_j) \le (-\widetilde{b}_j) \vee (\widetilde{\Delta} - \widetilde{a}_i - \widetilde{a}_j) \vee (-\widetilde{a}_i) .$$

Based on law (12), we take the minus sign outside all brackets in the last inequality and multiply its left- and right-hand sides by -1, which results in

$$\widetilde{a}_i \wedge \widetilde{b}_j \wedge (\widetilde{a}_i + \widetilde{a}_j - \widetilde{\Delta}) \le \widetilde{a}_j \wedge \widetilde{b}_i \wedge (\widetilde{a}_i + \widetilde{a}_j - \widetilde{\Delta}) . \tag{25}$$

The symbol $\wedge$ in (25) is conjunction (13). Let us solve inequality (25). We rewrite it in the form

$$\widetilde{L} \wedge \widetilde{D} \le \widetilde{M} \wedge \widetilde{D}, \qquad (26)$$

where $\widetilde{L} = \widetilde{a}_i \wedge \widetilde{b}_j$, $\widetilde{M} = \widetilde{a}_j \wedge \widetilde{b}_i$, $\widetilde{D} = \widetilde{a}_i + \widetilde{a}_j - \widetilde{\Delta}$. The logical inequality (26) for interval quantities is solved by the same separation method as for point quantities [7]. We obtain $\widetilde{L} \le \widetilde{M}$ (always), $\widetilde{L} > \widetilde{M}$ (for $\widetilde{D} \le \widetilde{M}$) for (26), and, on returning to the original quantities, we derive the following solutions to (25):

$$\widetilde{a}_i \wedge \widetilde{b}_j \le \widetilde{a}_j \wedge \widetilde{b}_i, \qquad (27)$$

$$\widetilde{a}_i + \widetilde{a}_j - \widetilde{\Delta} \;\le\; \widetilde{a}_j \wedge \widetilde{b}_i \;<\; \widetilde{a}_i \wedge \widetilde{b}_j, \qquad (28)$$

The inequality (27) involves only time characteristics of jobs i and j. If (27) holds then jobs i, j in the optimal sequence P_n follow in the order $i \to j$ irrespective of the order of the other jobs. Besides the characteristics of i and j, inequality (28) contains the parameter Δ depending on subsequence P_k preceding i and j. Fulfillment of condition (28) means that jobs i and j in the optimal sequence P_n for execution of jobs follow in order $i \to j$ only in the case when the preceding subsequence P_k has the corresponding value of the parameter Δ. It is clear that for optimal scheduling of jobs it is more advisable to use condition (27) stated as the following independent theorem.

Theorem 1. For jobs i and j in optimal sequence of execution of all n jobs in a two-unit nondetermined system with execution times of first and second operations of job i in form of intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$ to follow in the order $i \to j$ irrespective of the order of execution ofother jobs it is necessary and sufficient that the time parameters i and j satisfy condition (27).

5. Reduction to Deterministic Problems

We will reduce the optimality conditions for the order of execution of jobs in the nondeterministic engineering system in question that are established in Theorem 1 to the well-known optimality conditions for the order of execution of jobs in different deterministic systems [4]. Consider two two-unit deterministic systems. Let the execution times of the first and second operations on an arbitrary job i in the first system be equal to the lower bounds a_{i1} and b_{i1} of the times $\widetilde{a}_i$ and $\widetilde{b}_i$ of execution of these operations in given nondeterministic system, respectively, and let in other systems these times be equal to the lower and upper bounds a_{i2} and b_{i2} of the times $\widetilde{a}_i$ and $\widetilde{b}_i$. We will call these systems accordingly the lower and the upper deterministic boundary systems relative to the nondeterministic system.

Theorem 2. For jobs i and j in optimal sequence of execution of all n jobs in two-unit nondetermined system with execution times of first and second operations of job i in form of the intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$ to be carried out in the order $i \to j$ irrespective of the order of execution of the other jobs it is necessary and sufficient that jobs i and j be carried out in same order irrespective of execution of other jobs, i.e. in order of execution in the optimal sequences for execution of all jobs in two deterministic two-unit systems, namely in lower and upper boundary systems. Theorem 2 implies following theorem.

Theorem 3. For a type (1) permutation $P_n = (i_1,...,i_n)$ be an optimal sequence of execution of n jobs in a nondeterministic two-unit engineering system with execution times of the first and second operations on job i in form of intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$ it is necessary and sufficient that P_n be also the optimal sequence of the execution of n operations in the lower and upper boundary systems. Theorem 3 implies the two theorems below.

Theorem 4. The set M of all optimal sequences of n jobs in a nondeterministic two-unit computing system with execution times of the first and second operations of job i in form of intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$ is the intersection of the sets M_l and M_u of the all optimal sequences of n jobs in its lower and upper deterministic boundary systems.

Theorem 5. For an optimal secuence $P_n = (i_1,...,i_n)$ of execution of all n jobs to exist in a nondeterministic two-unit computing system with execution times of the first and second operations of job i in form of intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$ it is necessary and sufficient that the intersection of the sets M_l and M_u of all optimal sequences of the execution of n jobs in its lower and upper deterministic boundary systems be nonempty.

Theorems 4 and 5 imply the following direct solution algorithm for the stated problem, i.e. for finding an optimal sequence $P_n = (i_1,...,i_n)$ of execution of n jobs in a nondeterministic two-unit system with execution times of first and second operations of job i in the form of intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$.

Step 1. Finding the set M_l of all optimal sequences of execution of n jobs in lower boundary system of original system with execution times $a_i = a_{i1}$ and $b_i = b_{i1}$, which are the times of 1st and 2nd operations of job i. The well-known solution methods for deterministic two-stage problem of scheduling in industrial systems are used [2, 3, 5, 6].

Step 2. Finding the set M_u of all optimal sequences of execution of n jobs in upper boundary system of the original system with execution times $a_i = a_{i2}$ and $b_i = b_{i2}$, which are the times of 1st and 2nd operations of job i, using the same methods as in Step 1.

Step 3. Finding the intersection $M_l \cap M_u$ of the sets, which is the set M of all optimal sequences of execution of n jobs in the given nondeterministic two-unit system. If $M \neq \varnothing$ then any sequence $P_n \in M$ is desired optimal sequence of execution of n jobs. If $M = \varnothing$ then there are no such sequences.

The suggested direct solution algorithm for the problem requires exhaustion when determining the intersection of the sets M_l and M_u, and therefore it is efficient only for $|M_l| = |M_u| = 1$ or for $|M_l|$ and $|M_u|$ close to 1. In case $|M_l|$ or $|M_u|$ is large, the direct algorithm is ineffective, and it is necessary to pass to the application of decision rules making it possible to find an optimal sequence of execution of jobs in a nondeterministic computing system without exhaustion.

6. Construction of Decision Rules

Consider an arbitrary two-unit deterministic computing system with the times of execution a_i and b_i of the first and second operations of job i in the first and second units respectively. We split the set of jobs into first, second and third classes of jobs: $(a_i < b_i), (a_i > b_i)$ and $(a_i = b_i)$. Then the decision rules for finding optimal sequences of execution of all jobs in a system are based on the schedule presented in the Table 1. An arbitrary cell (p,q) of the table contains a condition under which two arbitrary jobs i and j (belonging to the p-th and q-th classes respectively) are placed in order $i \to j$ in optimal sequence. The schedule makes it possible to state a non-exhaustive decision rule for finding all optimal sequences of jobs for any set of jobs.

Table 1

Class of job	Order of execution		
	1	2	3
1	$a_i \le a_j$	Always	always
2	never	$b_i \ge b_j$	$b_i \ge b_j$
3	$a_i \le a_j$	Always	always

For example, the cell (1,1) shows that for the set of jobs of the first class the optimal execution sequence is obtained by arranging job i in increasing (more precisely, nondecreasing) order relative to parameter a_i.

Let us apply a similar approach to a given nondeterministic two-unit computing system with execution times of the first and second operations of job i in the form of intervals $\widetilde{a}_i = [a_{i1}, a_{i2}]$ and $\widetilde{b}_i = [b_{i1}, b_{i2}]$. Along with this system consider its lower and upper deterministic boundary processing systems (Table 1). The former has execution times a_{i1} and b_{i1} of the 1st and 2nd operations of job i, and for the latter these values are a_{i2} and b_{i2}. By Theorem 3 an optimal sequence of execution of jobs in a nondeterministic system is also an optimal sequence of the execution of jobs in its lower and upper deterministic boundary systems. Therefore, the optimality condition for a sequence of jobs in a nondeterministic system is the intersection of similar conditions for its lower and upper boundary systems.

Consider lower boundary system. In accordance with presented technique we split its set of n jobs into jobs of the first, second and third classes: $(a_{i1} < b_{i1}), (a_{i1} > b_{i1})$ and $(a_{i1} = b_{i1})$ respectively. Let us compile the schedule of execution for this system (see Table 2).

We now consider the upper boundary system. By the same technique we split its set of n jobs into jobs of the first, second and third classes: $(a_{i2} < b_{i2}), (a_{i2} > b_{i2})$ and $(a_{i2} = b_{i2})$. We thus obtain Table 3 of the schedule of operation of this system.

The schedule for a nondeterministic processing system is intersection of schedules of its lower (Table 2) and upper (Table 3) deterministic boundary systems of the original system. This table is compiled in the following way. Using the combination of some cells (p_l, q_l) and (p_u, q_u) of Tables 2 and 3 respectively we form the cell $((p_l, p_u), (q_l, q_u))$ of the desired table into which the condition equal to the intersection of the conditions in the cells (p_l, q_l) and (p_u, q_u) of Tables 2 and 3 respectively is inserted.

If the inserted condition in the cell contains the words «always» and «never» it is simplified in the following way: $A \cap \text{always} = A$, $A \cap \text{never} = \text{never}$, A is arbitrary.

Table 2

Class of job	Order of execution		
	1*l*	2*l*	3*l*
1*l*	$a_{i1} \le a_{j1}$	Always	always
2*l*	never	$b_{i1} \ge b_{j1}$	$b_{i1} \ge b_{j1}$
3*l*	$a_{i1} \le a_{j1}$	Always	always

Table 3

Class of job	Order of execution		
	1*u*	2*u*	3*u*
1*u*	$a_{i2} \le a_{j2}$	Always	always
2*u*	never	$b_{i2} \ge b_{j2}$	$b_{i2} \ge b_{j2}$
3*u*	$a_{i2} \le a_{j2}$	Always	always

The presented procedure is carried out for all possible combinations of cells in Tables 2 and 3. As a result schedule for nondeterministic processing system (Table 4) is constructed. In each cell $((p_l, p_u),(q_l, q_u))$ of the Table 4 the complex condition is presented under which the arbitrary jobs i and j (where the job i belongs to the p_l-th class of the lower boundary system and to the p_u-th class of the upper boundary system and job j belongs to the q_l-th class of the lower boundary system and to the q_u-th class of the upper boundary system) are placed in an optimal sequence of execution of jobs in the order $i \to j$. The conditions in Table 4 are given in the form of inequalities for the boundaries of intervals determining the execution times of jobs and, when possible, in the form of inequalities for the indicated intervals.

For construction of non-exhaustive decision rules for determining all optimal sequences of executions of jobs in nondeterministic systems we use Table 4. In contrast to deterministic systems an optimal sequence of execution of jobs in nondeterministic systems may not exist. This is due to the fact that different intervals (execution times of jobs) may not be compaprble and may not have minimal and maximal intervals. The decision rules for each set of classes of jobs forming the set of jobs performed in the nondeterministic system are constructed separately.

7. Example

We will construct the decision rule for finding the optimal sequences of the execution of jobs belonging to the single class $(1_l, 1_u)$. The condition in the cell $((1_l, 1_u),(1_l, 1_u))$ of Table 4 shows that the jobs i in the desired sequences must follow in nondecressing order of the interval parameter $\widetilde{a}_i = [a_{i1}, a_{i2}]$ or, which is the same, in nondecreasing order of the two parameters: a_{i1} and a_{i2}. What has been said implies the following rule: arrange all jobs i in nondecreasing order relative to the parameter a_{i1} and thus obtain the corresponding set M_1 of ordered sequences of jobs; arrange all jobs i in nondecreasing order relative to the parameter a_{i2} and thus obtain a similar set of sequences M_2; take the intersection of the sets M_1 and M_2 which gives the desired set of optimal sequences of jobs.

Table 4

Class of job	Order of execution								
	$1l1u$	$1l2u$	$1l3u$	$2l1u$	$2l2u$	$2l3u$	$3l1u$	$3l2u$	$3l3u$
$1l1u$	$\widetilde{a}_i \le \widetilde{a}_j$	$a_{i1} \le a_{j1}$	$a_{i1} \le a_{j1}$	$a_{i2} \le a_{j2}$	always	Always	$a_{i2} \le a_{j2}$	always	always
$1l2u$	never	$a_{i1} \le a_{j1}$ $b_{i2} \le b_{j2}$	$a_{i1} \le a_{j1}$ $b_{i2} \le b_{j2}$	never	$b_{i2} \ge b_{j2}$	$b_{i2} \ge b_{j2}$	never	$b_{i2} \ge b_{j2}$	$b_{i2} \ge b_{j2}$
$1l3u$	$\widetilde{a}_i \le \widetilde{a}_j$	$a_{i1} \le a_{j1}$	$a_{i1} \le a_{j1}$	$a_{i2} \le a_{j2}$	always	Always	$a_{i2} \le a_{j2}$	always	always
$2l1u$	never	never	never	$b_{i1} \ge b_{j1}$ $a_{i2} \le a_{j2}$	$b_{i1} \ge b_{j1}$	$b_{i1} \ge b_{j1}$	$b_{i1} \le b_{j1}$ $a_{i2} \le a_{j2}$	$b_{i1} \ge b_{j1}$	$b_{i1} \ge b_{j1}$
$2l2u$	never	never	never	never	$\widetilde{b}_i \ge \widetilde{b}_j$	$\widetilde{b}_i \ge \widetilde{b}_j$	never	$\widetilde{b}_i \ge \widetilde{b}_j$	$\widetilde{b}_i \ge \widetilde{b}_j$
$2l3u$	never	never	never	$b_{i1} \le b_{j1}$ $a_{i2} \le a_{j2}$	$b_{i1} \ge b_{j1}$	$b_{i1} \ge b_{j1}$	$b_{i1} \le b_{j1}$ $a_{i2} \le a_{j2}$	$b_{i1} \ge b_{j1}$	$b_{i1} \ge b_{j1}$
$3l1u$	$\widetilde{a}_i \le \widetilde{a}_j$	$a_{i1} \le a_{j1}$	$a_{i1} \le a_{j1}$	$a_{i2} \le a_{j2}$	always	Always	$a_{i2} \le a_{j2}$	always	always
$3l2u$	never	$a_{i1} \le a_{j1}$ $b_{i2} \le b_{j2}$	$a_{i1} \le a_{j1}$ $b_{i2} \le b_{j2}$	never	$b_{i2} \ge b_{j2}$	$b_{i2} \ge b_{j2}$	never	$b_{i2} \ge b_{j2}$	$b_{i2} \ge b_{j2}$
$3l3u$	$\widetilde{a}_i \le \widetilde{a}_j$	$a_{i1} \le a_{j1}$	$a_{i1} \le a_{j1}$	$a_{i2} \le a_{j2}$	always	Always	$a_{i2} \le a_{j2}$	always	always

8. Conclusion

In this article we give some theoretical facts -in the field of jobs sequences in the systems. They are touch some problems connected with uncertainty of time parameters of systems. It is shown that the problems can be reduce to complete determined case.

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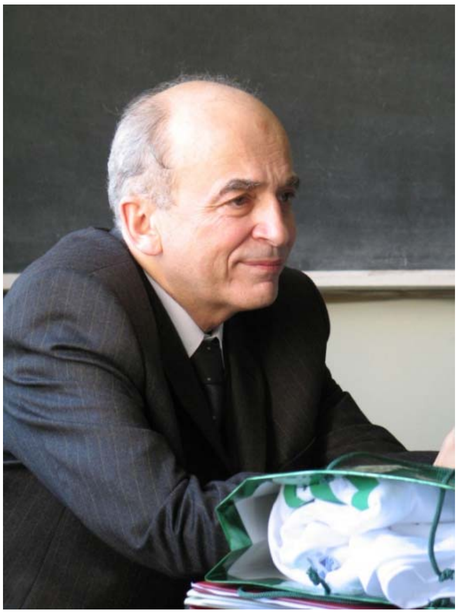
studia humana
QUARTERLY JOURNAL

Studia Humana
Volume 3:1 (2014), pp. 48—50
DOI: 10.2478/sh-2014-0005

Is Russian Mathematics Promising Still?



Semën Kutateladze is a Russian mathematician who has continued and enriched the scientific tradition of Leonid Kantorovich. He works at the Sobolev Institute of Mathematics of the Russian Academy of Sciences and Novosibirsk State University and known for contributions to functional analysis and its applications to vector lattices and optimization. In particular, he has made contributions to subdifferential calculus for vector-lattice valued function, to whose study he applied the methods of Boolean-valued models and infinitesimals.

Andrew Schumann: Soviet mathematics had one of the best traditions, and the results by Soviet mathematicians were accepted all over the world. Why did Soviet mathematics become so successful? The Soviet totalitarian system cannot be considered the best context for any scientific activity. Which fields of Soviet mathematics were best developed?

Semën Kutateladze: This is a topic for a special monograph, and so I answer superficially. We should distinguish between the terms “Soviet mathematics” and “Russian mathematics.” The latter implies mathematics that is produced by those who think in Russian. There was no Soviet language and mathematics in the USSR was part of Russian mathematics. No one calls Euler a Russian mathematician. Mathematics was successful in the USSR mainly by the special ideological role of science. Theoretically, socialism was viewed as a science, and scientists had privileges in the USSR. For instance, each full member of the Academy of Sciences had the salary that exceeded the salary of a member of the Political Bureau of the Communist Party. A gifted mathematician could become an academician in his prime years. So mathematics was a path to some freedom in the stale totalitarian atmosphere. Russian mathematics in the times of the USSR had contributed to practically all areas. It is sufficient to list the names of world-renowned late celebrities: N.N. Luzin, P.S. Novikov, A.N. Kolmogorov, S.L. Sobolev, A.I. Malcev, L.V. Kantorovich, A.A. Markov, A.D. Alexandrov, V.I. Arnold, O.A. Ladyzhenskaya, and many others.

Andrew Schumann: Is Russian mathematics of today is promising still?

Semën Kutateladze: In my opinion, it is still promising but the schools are being dispersed and mathematics is not the business that makes you free in modern Russia. So, the vistas of mathematics in Russia are dim.

Andrew Schumann: Grigori Perelman who has proved the Poincaré conjecture and published this result not in a journal, but in *arxiv.org*, an open e-print archive, is the most eminent Russian mathematician today. Now he is a best-known sample of maths genius with a strange and unexplained behaviour. How far typical is his behaviour for Russian mathematicians? Mukhtarbay Otelbaev, a mathematician from Kazakhstan, has published a 100-page paper on the existence of strong solutions for the Navier-Stokes equations. His paper is printed in a journal that is not indexed in Web of Science or anywhere else and it is not the best place for such a result. It is quite strange too. How many maths geniuses who do not satisfy common standards of social activity in science can we expect in the post-Soviet countries? Why do we face these situations?

Semën Kutateladze: According to the last evidences, Mukhtarbay Otelbaev does not have a correct proof. Grisha Perelman is not strange at all. Grisha had opened his results to the community for checking, claiming nothing. Grisha is a *champion of scientific ethics* and an exemplar of the highest moral standards. The majority does not meet the standards and considers Grisha a freak. History lists many analogous human follies.

Andrew Schumann: In Russia the wide-ranging structural reforms in science have been started recently. Are they promising? How can they change Russian mathematics? Can h-index and other tools used now in the Russian science measure both the productivity and impact of the published work of a mathematician?

Semën Kutateladze: The reforms of science in Russia are conceived and implemented by professional reformists *per ce.* Those are bureaucrats who agree with nobody but themselves. The practical dissolution of the Russian Academy of Sciences will hamper science in Russia for decades. As regards the h-factor and its next of kin, suffice it to say that the International Mathematical Union has appealed to abstain from bibliometric indices in making any decision on the contribution and status of a fellow mathematician.

Andrew Schumann: You are both a mathematician and logician. The program of Berlin and Vienna Circle (the so-called Hilbert's program) as well as the program of Lvov-Warsaw School consist in reducing mathematics to logic. But this program became unsuccessful for different reasons. Do you think that a new program of reducing mathematics to logic is possible yet, e.g. are new logical tools possible in infinitesimal analysis?

Semën Kutateladze: Mathematics became logic in the twentieth century. But logic is understood today in a much broader context than in the times of the battle for the ultimate foundation which is viewed now as a wild-goose chase. Logic was a dogma yesterday. Logic is the fortress of freedom today. As regards new logical tools, these are galore not only in infinitesimal analysis.

Andrew Schumann: What is mathematical knowledge? Do mathematical objects exist? What are infinities?

Semën Kutateladze: Those are insurmountable questions and so my answer will be trivialities. Mathematical knowledge is a collection of very simple universal intellectual patterns. Our ancestors differed a cave from a hole − that is topology; they used cardinal and ordinal counts − that is set theory and algebra; they sought for trend and predict future − that is calculus and probability. We safeguard and develop their techniques. Mathematical objects are figments of thought. There are no logarithms, nor Lie groups without humans. But we are humans and we use these figments. An infinite is a number greater than any assignable number. For instance, the number of molecules in the chair I sit now. The modern details of this ancient definition are revealed in the Robinsonian infinitesimal analysis.

Andrew Schumann: Please say about problems or theorems you are working on?

Semën Kutateladze: My current interests in mathematics consist mainly in finding some formalities that will unify the nonstandard models of set theory and simplify their use in analysis. I also dream of a new variational calculus suitable for multiple criteria optimization problems.