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Preface. Conditional:
Conceptual and Historical Analysis

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Abstract: The logic of conditional is developed hereby in a series of papers, contributing to a historical and critical analysis of what the logical constant is expected to mean.
Keywords: assertion, commitment, conditional, implication, connection, Frege-Geach argument, relevance.

Conditional is certainly one of the most controversial logical constants. Even a radical logical pluralist will concede this point, despite the relative meaning of the so-called "syntecategorematics" in natural language, there are not so many writings about the ambiguity of conjunction or disjunction than about conditional. This is mainly due to the famous ambiguity between object language and metalanguage, famously noted by Willard van Orman Quine and focusing on the close similarity between conditional and deduction. The same confusion may be mentioned about the constant of negation, all the more that the problematic meaning turns out to be also related in the characterization of conditional. Admittedly, it is not an easy task to clarify the debate evolving around what is currently dubbed as either "conditional" or "implication". A number of conditionals occurred in the contemporary history of logic: material strict, intuitionistic, relevant, linear, and so on. Whether classical or non-classical, a common difficulty comes from the nature of the "nexus" between the antecedent and the consequent. Should there be a causal, temporal, or merely casual relation between them?

The present issue does not want to give an exhaustive survey of the literature having to do with conditional. However, some of the most renown problems of logic are discussed: the paradoxes of material implication, to the effect that a conditional is true whenever its antecedent is false; Russell's "Embedding Problem" (or the Frege-Geach Problem), which deals with the status of assertion and the troublesome role of antecedent in conditional statements; the problem of pure implication, i.e., how to find a proper characterization of conditional which makes no use of other logical constants in its definition and makes it differ from the other ones.

In order to disentangle such a thorough discussion around conditional, the present issue intends to bring its own contribution to the debate around what conditional means. For this purpose, this special issue proposes a twofold reading of the logical constants in its conceptual and historical aspects. Two famous inference rules, Modus Ponens and the Deduction Theorem, cannot be neglected in such an enterprise.

In “The Semantics and Pragmatics of Conditional in al-Farabi and Avicenna”, Saloua Chatti proposes a comparative analysis between the logical analyses of conditional by al-Farabi and Avicenna. Beyond a various taxonomy in their respective writings, it clearly appears that a number of common features emerge from the account of conditional in such Arabic medieval commentaries. Thus temporal, causal, and strictly logical connections between antecedent and consequent are clearly mentioned therein, thus showing that what still prevails in the modern literature around conditional was already emphasized in both Aristotelian and post-Stoician commentators.

In “Implications and Limits of Sequences”, Alexandre Costa-Leite and Edelcio Souza challenge the existence of essential properties for conditional in natural language. Accordingly, the historical aspects of conditional are kept aside from this paper. Rather, the authors favor a normative approach over the descriptive one and lead to an original definition of conditional in mathematical terms of limits in a finite sequence of sentences. The latter is promoted as a more promising account than the truth-functional, lattice-theoretical, or structural one (by Arnold Koslow).

In “Assertions and Conditionals: A Historical and Pragmatic Stance”, Daniele Chiffi and Alfredo Di Giorgio combine a more recent historical approach with a modern formal language of pragmatics. Thanks to a thorough survey of medieval philosophers and logicians including Abelardus, Ockam, or Bricot, the authors want to show that a serious “assertion candidate” requires an introduction of two additional concepts, viz. assertion and judgment. The illocutionary import of conditional thus leads to a formal logic, Logic for Pragmatics, where the distinction between radicals (propositional contents with no assertive force) and sentences (asserted propositions) helps to redefine conditional in close connection with intuitionistic and modal logics.

In “Conditionals in Interaction”, James Trafford also advocates the assertoric import of conditional, by attempting to clarify the distinction between the assertion of a conditional and a conditional assertion. After reminding the objective reading of assertion within the antirealist literature, the author defends a social view of assertion within a social community of speakers: in the line of Brandom’s view of logic as a “game of giving and asking for reasons”, conditional should better be understood as a whole hypothetical move rather than a categorical commitment performed by a speaker upon the antecedent.

In “The Football of Logic”, Fabien Schang follows Trafford’s dynamic characterization of conditional in a game-theoretical description of proof. More than that, the social aspect of logic is emphasized by an analogy between logic and football. Through the Tarskian framework of logic as truth-preservation, Schang argues that the players of the two kinds of game equally purport to preserve something whilst aiming at a goal against another player. A “strong” version of conditional finally results from a structured interpretation of truth-values, both erasing the paradoxes of material implication and adapting the dialogical question-answer game into a four-valued algebraic logic.

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The Semantics and Pragmatics of the Conditional in al-Fārābī's and Avicenna's Theories

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Abstract: In this paper, I examine al-Fārābī's and Avicenna's conceptions of the conditional. I show that there are significant differences between the two frames, despite their closeness. Al-Fārābī distinguishes between an accidental conditional and two "essential" conditionals. The accidental conditional can occur only once and pragmatically involves succession. In the first "essential" conditional, the consequent follows regularly the antecedent; pragmatically it involves likeliness. The second "essential" conditional can be either complete or incomplete. Semantically the former means "if and only if"; pragmatically it means "necessarily if and only if". The latter is expressed by 'if, then' and means entailment; pragmatically, it involves necessity and the inclusion of the antecedent into the consequent. As to Avicenna, he rejects explicitly al-Fārābī's complete conditional and distinguishes between the *luzūm* (real implication) and what he calls *ittifāq*. He quantifies over situations (or times) to express the various conditionals. The two universals $\mathbf{A}_C$ and $\mathbf{E}_C$ are expressed by "In all situations, if…, then…", while the two particulars $\mathbf{I}_C$ and $\mathbf{O}_C$ are expressed by "In some situations, if…, then..". This gives them a modal connotation, and makes the universals close to strict implications. Pragmatically, $\mathbf{A}_C$ presupposes the truth of the antecedent, but there is no such presupposition in $\mathbf{E}_C$, while what is presupposed in both $\mathbf{I}_C$ and $\mathbf{O}_C$ is a (possible) conjunction.
Despite these differences, in both systems, the conditional is not truth functional, unlike the Stoic conditional.

Keywords: Essential *vs* accidental conditional, *ittifāq*, *luzūm*, entailment, strict implication, strict equivalence, quantified conditionals.

1. Introduction

In their respective hypothetical logics, al-Fārābī (873-950, AD) and Avicenna (980-1037, AD) present syllogisms containing conditional and disjunctive propositions. These syllogisms express explicitly and implicitly their conceptions of these connectives.

In the following paper, we will focus especially on the meanings of conditionals and will try to answer the following questions: How do both philosophers define the connective of conditional? How do they use this logical constant in their theories? Do these definitions contain a pragmatic aspect, given that pragmatics deals with presuppositions, contexts, intentions and implicit meanings that go beyond what is literally said? Are there any differences between the two theories?

In answering these questions, we will clarify the two conceptions defended by these authors and the differences between these conceptions which are made explicit through the analysis of their respective hypothetical syllogisms and the rules admitted in both frames.

2. The Meaning(s) of Conditionals in al-Fārābī's Frame

Before starting the analysis of al-Fārābī's doctrines about conditionals, let us first clarify the words 'semantic' and 'pragmatic'. As it is usually defined, semantics has to do with the explicit meanings of words and propositions. These meanings help determine the truth-values of sentences. While pragmatics has to do with contexts, presuppositions, implicit meanings and the intentions of the utterers. It is by taking into account these presuppositions, intentions and contextual circumstances that the truth-values of sentences can be settled.

Now we don't find any explicit definition of semantics and pragmatics in al-Fārābī's or Avicenna's texts, but the examples they provide in their respective hypothetical logics are based implicitly or explicitly on semantic and pragmatic considerations. This is why it makes sense to talk about the semantics and pragmatics of the logical conditional in their respective frames. Let us start by al-Fārābī.

Al-Fārābī's hypothetical logic is presented in his *al-Qiyās* (the counterpart of *Prior Analytics*) and *al-Maqūlāt* (the counterpart of the *Categories*). The hypothetical syllogisms contain either conditional propositions or disjunctive ones. In these two treatises, conditionals as well as disjunctions are defined and considered in the context of the valid hypothetical syllogisms. These valid syllogisms are developed mainly in *al-Qiyās*.

In *al-Maqūlāt*, the conditional proposition (*mutalāzima*) is such that "if one of its elements exists, the other one exists too by means of that [*idhā wujida aḥaduhumā, wujida-al-'ākharu bi wujūdihi*]" [4, p. 78]. There is thus a dependence relation between the antecedent and the consequent of the conditional. But this relation is not exactly the same from one kind of conditional to another.

There are basically three kinds of conditional propositions:

1. The *accidental* conditional, where the consequent and the antecedent are related by accident (*bi-l-'araḍi*). As an example, al-Fārābī provides the following: "If Zayd comes, 'Amr leaves" [2, p. 127]. Here, the consequent may follow the antecedent, but not always, nor often. But 'follows from' may just mean a succession in time. We can note, however, that the two verbs 'come' and 'leave' are semantically related.

2. The *essential* conditional, which is of two kinds:

2a. A first kind of conditional where the consequent follows the antecedent "most of the time" [2, p. 127] but not always. This is illustrated by the following example: "When Sirius rises in the morning, the heat will be severe and the rains will cease" [2, p. 127, translation Wilfrid Hodges [10, p. 247.]

2b. A second kind of conditional where the consequent *necessarily* follows from the antecedent. In this case, the consequent always follows the antecedent.

The last conditional is either *complete* or *incomplete*. The complete one is a biconditional (or an equivalence): necessarily when the antecedent holds, the consequent holds, and conversely. This kind is illustrated by the following sentence:
$2b_1$: 'If the sun rises, it is daytime' (and vice versa)
As to the incomplete one, it is a single conditional which does not convert. For instance, we can say:
$2b_2$: 'If this is a human, it is an animal' (but not conversely) [4, p. 78].

The second example looks like a strict implication, while the first one looks like a strict equivalence. Both involve a semantic and necessary link between their elements, which might also be causal. The semantic aspect is related to the explicit meanings of the antecedent and the consequent. Given these meanings, the conditional relation holds.

Note that even in (1) and (2a), there is either a semantic or a causal link, for in case (1), the words 'comes' and 'leaves' are semantically related, while the events evoked in (2a) happen successively and might be related causally, even if there is no necessity in the link between the antecedent and the consequent. The two implications are thus intensional because of the semantic relations between the antecedent and the consequent. However, these semantic relations cannot determine alone the truth-values of the sentences, for in sentence (1), for instance, there is no necessary link between the coming of Zayd and the leaving of 'Amr; so if the sentence is true, its truth is not due to the meanings of the words 'coming' and 'leaving'; rather it would be due to the facts that really happened. So this kind of conditional is not comparable to those expressed by sentences which are "true solely by virtue of the meanings of the words" as Carnap characterizes them. In other words, this conditional is not "analytically true" in the Carnapian or modern sense, despite the semantic link between the antecedent and the consequent. Rather the truth of the whole conditional has to do with the context of utterance of such a sentence, hence it has also a pragmatic aspect.

As to sentence (2a), its alleged "essential" character raises a problem, for if the consequent follows the antecedent only "most of the time" but not always, how could the relation be "essential"? What does the word "essential" mean in that particular case, given that "essential" is usually connected with the notion of necessity which is stronger than what seems to be involved here? This particular case seems strange to Wilfrid Hodges too, who says in his book *Mathematical Background to the logic of Avicenna* (2016): "Curiously he allows that some 'essential' relations hold only for the most part" [10, p. 248].

A possible answer would be to interpret (2a) as expressing some kind of *natural* connection, i.e. a connection that holds in nature and can be observed most of the time. "Essential" would then be related to the context of utterance of the sentence, since we cannot consider that the linguistic meanings alone make the sentence truth. We can perhaps also evoke some kind of "non-technical" or "broad" meaning of the words "essential" and "essentially" which makes them close to "mostly" as one reviewer suggests. In that case, the word "essential" does not require necessity; it would have to do with the fact that the two elements are fundamentally, though not necessarily related. As a matter of fact, al-Fārābī's analysis relies on ordinary meanings of words in Arabic and this could explain and justify some of what he says about the conditional operator. Here too, we can also evoke a pragmatic aspect, which results from the consideration of the contextual circumstances and facts that make the sentence true.

The relations in (1), (2a) and (2b) are different, for there is no necessity and no reciprocity in (1) and (2a), while in (2b) the relation is clearly symmetric, given that what is presupposed by al-Fārābī is a biconditional, although he does not use a specific (and different) word to name it. What then, is meant by 'implication' in (1) and (2a)? In particular, given its "accidental" character, can we say that (1) is a material conditional? The answer is: No. First because the material conditional is extensional since there is no semantic link between p and q, which may be entirely independent semantically, while (1) is intensional, for it depends on the meanings of its elements. Secondly because there seems to be a temporal succession between the antecedent and the consequent, which is not always the case in the material conditional. Thirdly because al-Fārābī does not give the whole truth conditions of this conditional, since the cases where the antecedent is false are not sufficiently clarified, as we will see below.

What about (2a)? Unlike (1), which is said to be "accidental" by al-Fārābī, for it may happen just once, (2a) expresses a regular and frequent succession in time. But this regularity does not mean that the relation between the antecedent and the consequent always holds. For this reason, the relation lacks necessity. So it cannot be a strict implication. However, this link is not accidental either, for it is natural, observable and mainly empirical. We could say that it stands between "necessary" and "accidental" just as "general" stands between "universal" and "particular" and "most" stands between "every" and "some" in ordinary languages. This also means that in that case too, the relation is not a material conditional. It could be expressed by means of a probable relation between the antecedent and the consequent as follows: 'If p then probably q', or 'Probably (if p then q)', where 'p' and 'q' are semantically related. But such a kind of probability is better expressed by the word "likely" and is not comparable to the standard and mathematical account of probability. Rather it is more like some kind of *imprecise* probability.

In both cases, the meanings explicitly carried are different, despite the use of the single expression "if…then", for in (1), the relation between the antecedent and the consequent seems to be contingent or chancy, while in (2a), this relation is not chancy ; rather it is regular and likely to occur, even if it is not really necessary.

As to presuppositions, they are also different, for what is *presupposed* in (1) seems to be a *possible* implication involving a succession in time, since the antecedent precedes the consequent, but not conversely, while (2a) seems to implicitly express a *probable* implication, despite the fact that al-Fārābī does not use an explicit modal word in that context. It also involves a succession in time, which is regular and considered as "essential" by al-Fārābī. This succession is empirical and suggests a natural link between the antecedent and the consequent, even if this link is not necessary. Possibility or probability are thus more implicitly suggested by the examples chosen than explicitly expressed. They may be part of the *pragmatic* meaning of these kinds of conditionals, which convey different presuppositions related to the expression "if… then".

As to (2b), it is said to be necessary, whether it is complete or incomplete. The semantic meaning of "if…then" in the incomplete case is a non-convertible implication, where the antecedent always precedes the consequent, while the meaning of the complete implication is rather equivalent to the meaning carried by "if and only if". In both cases, what is presupposed is a necessary link between the antecedent and the consequent. So what is *pragmatically presupposed* in ($2b_1$) is "necessarily if […] then […]", while the presupposition conveyed by ($2b_2$) is "necessarily […] if and only if […]".

In both cases, the expression explicitly used by al-Fārābī is "if … then", but he does say that the complete meaning is "convertible", which means that the double implication is explicitly assumed, even if there is no additional *word* translating the convertibility. This kind of '[bi] conditional' may be rendered by the following complex expression: "If p then q and if q then p" which is strongly suggested in the arguments provided by al-Fārābī.
This essential *complete* meaning validates the following syllogisms:

1. 'If p then q; but p; therefore q'

2. 'If p then q; but q; therefore p'

3. 'If p then q; but ~p; therefore ~q'

4. 'If p then q; but ~q; therefore ~p' [4, p. 79, my formalization]

In these hypothetical syllogisms, what is presupposed is 'if and only if' rather than simply 'if…then', as appears in the following quotation: "And those expressing a complete implication are those where if *whatever* element holds, the other one necessarily holds too by means of it (*bi* *wujūdihi*), for if the first one holds, the second one necessarily holds, and if the second one holds, the first one necessarily holds too.” [2, p. 127, my emphasis].

The example illustrating this case is the famous Stoic example: “If the sun is up, it is daytime”. Given the inseparability of both events, they always hold together, which justifies the completeness of the implication. Note that the same example expresses a simple (and truth-functional) conditional in the logic of the Stoics, according to Suzanne Bobzien, who says that the conditional is expressed in Stoic logic by means of the following negated conjunction: ‘Not (p and not q)’ [8, § 5.3]. This makes the Stoic account of the logical conditional different from that of al-Fārābī, since it does not contain any kind of modality. On the contrary, al-Fārābī’s necessary kind of implication would be closer to what is now called “strict implication”, which is expressed by “Necessarily (If p then q)”, although al-Fārābī does not use its equivalent formulation “Necessarily not (p and not q)” or “Impossibly (p and not q)”, given that he does not use a conjunction to express the conditional operator. The complete meaning of that implication, which validates the inferences 1-4 above and is illustrated by the classical example “If the sun is up, it is daytime”, would be more like a necessary biconditional, which we could express by the following “Necessarily (If p then q) and (If q then p)”.

As to the incomplete conditional exemplified by ($2b_2$), it admits the following syllogisms:

(1). ‘If p then q; but p; therefore q’

(2). ‘If p then q; but ~ q ; therefore ~ p’ [4, p. 79].

(1) corresponds to the *Modus Ponens*, while (2) corresponds to the *Modus Tollens*.
But it does *not* validate :

- ‘If p then q; but ~ p, therefore ~ q’

- ‘If p then q; but q; therefore p’ [5, p. 138]

given that the relation is not convertible.
The rejection of these two cases means that from the falsity of p, one cannot deduce the falsity of q, and that from the truth of q one cannot deduce the truth of p. So, when p is false, q could be either false or true. Similarly, when q is true, p could be either true or false.

In addition, he rejects the case where the antecedent is true and the consequent false, for when the antecedent is true, the consequent cannot be false.

In ($2b_2$) [‘If this is a human, it is an animal’] which illustrates this kind of incomplete implication, what is presupposed is the *inclusion* of the antecedent into the consequent for in the example provided, the class of humans is part of the class of animals.

So (1), (2a), ($2b_1$) and ($2b_2$) do not have the same semantic meaning nor the same pragmatic meaning. For in (1), there is probably only a succession of events, which may happen only once, while in (2a) this succession is regular and frequent, and in ($2b_1$) the link is causal while in ($2b_2$), it is an inclusion, hence it is conceptual. Here, the example chosen [‘If this is a human, it is an animal’] seems to pertain more to the Aristotelian categorical logic than to the hypothetical (or propositional) logic, for this sentence is another way to say that “All humans are animals”. This is why we could use the notion of inclusion to characterize the relation between the antecedent (the subject of the categorical sentence) and the consequent (the predicate of the categorical sentence). This means that the universal categorical sentence of the form **A** is expressible by a conditional in al-Fārābī’s frame. But in another book entitled “*al-Alfāḍ al-mustaʿmala fī al-mantiq*” (*The expressions used in logic*), al-Fārābī uses the word “inclusion” too, when talking about the hypothetical connected sentences which start by “If” or “whenever” or “when” and says that the antecedent in these sentences includes (*yataḍammanu*) the consequent, for he says that in the sentence “If the sun is up, it is daytime” “The rising of the sun includes (*taḍammana*) the succeeding emergence of the day" [1, p. 54], given that both events are closely related and that there is no daytime without rising of the sun. So although al-Fārābī distinguishes between the hypothetical sentences and the categorical ones, the notion of inclusion is involved in both kinds of conditional sentences, i.e. the hypothetical ones and the categorical ones. The consequent is included in the antecedent of a hypothetical connected sentence, as the predicate is included in the subject of the universal categorical one. This closeness will be acknowledged by Avicenna too as we will see in what follows.

Can we say that this conditional is truth-functional, given the truth-conditions provided by al-Fārābī, as is the case with the Stoics' conditional?

As a matter of fact, al-Fārābī provides two cases where a conditional is true and where the truth-values of either its antecedent or its consequent are deductible, i.e. known with certainty. These cases are the *Modus Ponens* and the *Modus Tollens* above. But does this mean that one can determine the truth-value of this conditional operator starting from the *values of its elements alone*? In other words, is the logical conditional truth-functional in his frame?

If we consider the examples given, there is always a natural or a semantic relation between the antecedent and the consequent. So the truth of the conditional operator depends also on the meanings of its elements, which are crucial to determine its truth or its falsity. As to the restrictions provided by al-Fārābī, they mean that it *is possible* for a conditional to be *true* when its consequent *alone* is true and also when its antecedent *alone* is false. But is the truth of this operator *warranted* in these two cases?

If we consider the examples provided, could we say, for instance, that sentence (1) ["If Zayd comes, 'Amr leaves"] is true if its antecedent is false, i.e. if Zayd does not come? Nothing indicates that this truth is warranted, nor even seriously considered by al-Fārābī.

As to sentence (2a) ['When Sirius rises in the morning, the heat will be severe and the rains will cease'], which involves a regular but not necessary link between the antecedent and the consequent, the truth of the sentence seems to presuppose the truth of its antecedent, since the case where the antecedent is false is not intuitively a case of truth for the whole conditional (or implication). Why should we say that this particular implication is true when its antecedent is false, i.e. when 'Sirius does not rise in the morning'? So, this kind of implication does not seem to be truth-functional, given that its truth-value depends on our intuitions and on the facts involved, not only on the truth-values of the elements. As we know, in the intuitive and ordinary sense, the implication or conditional does not seem to be true when its antecedent is false. In that case, its 'truth' would be very counter-intuitive. So the meaning of this kind of conditional is not determined by its truth conditions.

What about the necessary versions of the conditional, which are either complete or incomplete? The incomplete case is exemplified by the sentence 'If this is a human, then it is an animal'. It involves the notion of inclusion (to the class of humans inside the class of animals), which means that the sentence is true if such an inclusion holds, and false if it does not. But what happens if there are no human beings, that is, if the antecedent is false or in other words if the class of humans is empty? Would the sentence be true in that case? If we consider the fact that this sentence is an instance of the universal proposition 'Every human is an animal' [since we can express it by saying 'if *a* is a human, then *a* is an animal'] and if we take into account the fact that the universal affirmative cannot be true if its subject does not exist, in al-Fārābī's frame, as he says in *al-Maqūlāt*, where talking about the affirmative propositions, he claims that "when their subject does not exist, they are all false" [2, p. 124.13-14], then we may consider that the conditional proposition whose antecedent is false would not be true in al-Fārābī's theory. At least, its truth is not warranted. Consequently, this kind of conditional is not presumably truth-functional. If we compare al-Fārābī's position with modern ones, we can see that it is different from that of Strawson, a contemporary author who says that the sentence *lacks* a truth value whenever its subject does not exist as appears in what follows: "The more realistic view seems to be that the existence of children of John's is a necessary precondition not merely of the truth of what is said but of its being either true or false" [15, p. 174]. Unlike Strawson, who endorses the position that all sentences whose subjects are non-existent *do not have a truth value*, al-Fārābī says that this kind of sentences are *all false*. In this respect, his position is more Russellian than Strawsonian, despite the important differences between Russell's theory about the logical conditional, which is extensional, formal and mathematically expressible, and al-Fārābī's one.

What about the complete one? This is a necessary equivalence and the four valid moods involving it provided by al-Fārābī show that a [bi]-conditional is true when both its elements are true and when both its elements are false. So we know that this [bi]-conditional is true when its two elements have the same value. Consequently, we can deduce that it is not true when their values are different. The truth conditions of the complete implication seem thus to be settled. However, the very reason why these truth conditions are given depends ultimately on the meanings of the elements considered. So, here too, the truth-functionality of the [double] implication is not really clear.

What about Avicenna's theory on the conditional? This will be examined in the next section.

3. The Meaning(s) of Conditional(s) in Avicenna's Theory

According to Avicenna, conditionals may be weak, medium or strong. All these kinds of implication are expressed by specific and different words in the ordinary language (currently the Arabic language). The strong implication is the one expressed by '*in...fa*' (= if...then); the weak implication is expressed by '*matā*' (= when), while the mediate one is expressed by '*idhā*' (= if) [6, p. 235]. This classification suggests that these words carry different significations and do not involve the same presuppositions. For instance, the word '*matā*' (when) clearly has a temporal connotation, which may not be present when one uses the particle '*in*' (if). Avicenna also uses in some kinds of conditionals the word '*kullamā*' (= whenever) which evokes universality.

In his informal analysis of the logical conditional, he evokes al-Fārābī's distinction between a complete implication and an incomplete one without endorsing it, but his discussion of the matter clarifies the difference between the two conditionals involved and their nature in the complete implication. The complete implication is defined as: 'if p then q and conversely' (symbols added) and illustrated by 'whenever the sun rises it is daytime and whenever it is daytime, the sun rises' [6, p. 232]. As we can see, the two elements in the complete implication are clearly expressed and separated. In both cases, the link between the antecedent and the consequent is causal, but the causal relation occurs differently. For when one says: (1) 'If the sun is up, it is daytime', he means that 'the sun is the *cause* of the daytime', while when he says (2) 'if it is daytime, then the sun rises', he expresses rather the *inseparability* of the cause and the effect [6, pp. 233.17-234.1].

So in case (1), the word 'if' means that the antecedent is the *cause* of the consequent which follows from it for this reason, but in case (2), it means that they are (necessarily?) *concomitant*, given that the antecedent is not really the cause of the consequent. The conditionals involved carry then different meanings in both cases. In (1), the condition seems to be necessary, while in (2), it is rather a sufficient condition. The first condition is strong, while the last one is rather weak. There is thus a kind of asymmetry between both elements of the complex relation.

Note that, even in the modern sense, 'if and only if' contains the same asymmetry, for 'only if' is stronger than 'if', since it expresses a necessary condition.

But al-Fārābī's distinction is criticized by Avicenna who says that the complete implication does not respect the syntactical structure of the conditional, which is shown by the fact that the antecedent *precedes* the consequent in all cases. For saying that an implication might be convertible in some cases is like saying that the copula in an affirmative universal proposition might express an identity in some cases. But according to Avicenna, this is not compatible with the formal character of logic and is not the way al-Fārābī himself, as all traditional logicians, treats the universal affirmative in his syllogistic, given that he never considers the difference between a case where "the predicate is equivalent to the subject" and the case where it is not, when dealing with the universal affirmatives in the context of his discussion of the syllogistic moods. Avicenna observes that if the subject and the predicate were convertible in the universal affirmative, then *Darapti*, for instance, would have a universal conclusion, for we would say "If the predicate is equivalent to the subject, then [the proposition] would convert as itself; and if it is not, then it will convert as a particular" [6, p. 392]. But nobody, including al-Fārābī, ever considers this case or admits a third figure mood where the conclusion is a universal affirmative proposition following from two universal affirmative premises. Given this fact, one should take into account the form of the propositions in the hypothetical logic too.

The weak implication is illustrated by the following example: 'If every man is speaking, then every horse is whinnying' [6, p. 268]. Here, there is no real semantic relation, but only a kind of (contingent?) concomitance, for the antecedent cannot be said to be the cause of the consequent; they are rather independent, given that each of the two propositions may be true on its own, without being really *entailed* by the other one. This example illustrates what Avicenna calls '*ittifāq*', a word translated as 'chance connection' by Nabil Shehaby. This translation expresses the idea that the link between the antecedent and the consequent in an '*ittifāqī*' conditional is accidental; it is not strong, not natural and may be present only once. However, there is no consensus about the interpretation of the word '*ittifāq*', for it is understood in a different way by other commentators. For instance, Wilfrid Hodges says that the notion of chance (or of accident), evoked by N. Shehaby, is not relevant and does not account for the examples provided by Avicenna. According to him, '*ittifāq*' should be translated by the word 'agreement', which is more in accordance with the examples used by Avicenna. Agreement may be understood as an agreement with the facts, i.e. with what happens in the real world. This interpretation could be found in Wilfrid Hodges' recent book entitled *Mathematical Background to the Logic of Avicenna* (2016). It is supported by many examples like the one above [6, p. 268], since the whinnying of horses cannot in any sense be said to *depend* on the fact that men are speaking. This is what Avicenna calls "agreement in the truth" ("*al-muwāfaqa fī al-ṣidqi*") [6, p. 265.11]. In this sense, one might consider that '*ittifāq*' expresses in some way a conjunction rather than a conditional, since its elements do not depend on each other and since they are both true. Thus, *ittifāq* would express the weakest meaning of the conditional, since it does not involve any kind of dependence between the antecedent and the consequent. It would thus be close to (1) in al-Fārābī's classification. But Avicenna provides also other kinds of examples to illustrate *ittifāq* (or *muwāfaqa*) in which *only the consequent* is true, for instance, the following: "If every donkey is talking then every man is talking" [6, p. 270], and he even says that "Agreement (*muwāfaqa*) is nothing but (*laysa illā*) the configuration in which the consequent is true (*wa al-muwāfaqa laysa illā nafsu tarkīb al-tāli 'alā annahu haqqun*)" [6, p. 279.15]. So the question is the following: Can one interpret *ittifāq* as a conjunction? If not, what is its real logical meaning? We find an answer to this question in Wilfrid Hodges' article "Ibn Sīnā's propositional logic" (2014), where the author says that the interpretation of the proposition 'If p then q', where 'if…then' expresses an *ittifāq* may "come from Peripatetic speculations about how we can know that a sentence 'If p then q' is true. Two suggestions were: (a) We can know it because we know that q is true; (b) We can know it because we deduce q from p. Ibn Sīnā reads the *ittifāqī* case as (a) and the *luzūmī* case as (b)." [11, slide 25]. But he says that this notion is "strictly not logical at all" [11, slide 25]. This last judgment may be justified by the fact that when interpreted as true *only* when the consequent is true, it does not correspond to a conjunction, which is true only when *both* propositions are true, nor even to the usual material conditional, which is true in two other cases, besides the one considered by Avicenna.

The notion of *ittifāq* is distinguished from the *real* implication which is called *luzūm* [14, p. 37]. For in the real implication (*luzūm*), the consequent really *follows* from the antecedent either causally or semantically, while in the *ittifāq*, the link between the antecedent and the consequent is not causal. In sum, the real implication or *luzūm* involves the idea that the truth of the consequent *depends on* that of the antecedent, given that the consequent is true *only because* the antecedent is true, while there is no such dependence in the *ittifāq*, since both propositions can be true independently of each other, only by means of their agreement with reality. What Avicenna calls '*luzūm*' expresses thus some kind of *entailment*. In this sense, it may be true even if *both* its elements are *false* provided that the consequent *follows from* the antecedent.

The non-convertible implication may also be expressed by the word "whenever", which means 'in all cases' or 'in all situations' or 'in all times'. In that case, implications would convey a universal content as in 'Whenever (*kullamā kāna hāḍha*…) this is a man, it is an animal' [6, p. 232]. The implicit relation, here, is clearly an inclusion, which is not casual, but rather necessary.

In addition, Avicenna also mentions the counterfactual conditional which can be used to deduce the impossible from the impossible, as in 'If humans were not animals, they would not be sensitive' [6, p. 238]. This presupposes a necessary semantic relation between the antecedent and the consequent, for being sensitive implies being an animal, so that if one is not an animal, one cannot be sensitive. Pragmatically, as in all counterfactual conditionals, the antecedent is clearly *presupposed* to be *false*, and the consequent is what follows from that falsity. This makes this kind of conditional different from the indicative ones, where the antecedent might be false, undetermined or true, depending on the sentence. Despite this difference, however, it seems that in this kind of conditional too, there is clearly a dependence relation between the antecedent and the consequent. In this respect, its contrapositive form is also valid, for "If humans are sensitive, they are animals" is true too.

Generally speaking, conditional (or implication) is not truth-functional in Avicenna's frame, for he does not provide its whole truth conditions. The only settled cases are the ones where the antecedent is true. In that case, if the consequent is true, the implication is true, while if the consequent is false, the implication as a whole is false. In the cases where the antecedent is false, the implication *may* be true, but its truth is not warranted. When both propositions are false, a conditional or implication may be true but not necessarily in all cases, for it is the semantic link between its elements that makes it true, despite their falsity, for instance, when one says "If this is a stone, then it is inert", the sentence is true because being a stone implies being inert, whether the thing is really a stone or not. When the truth-values of the propositions are not known, a conditional may also be true, but not perforce in all cases. The example provided by Avicenna is the following: "If Abdullah is writing, then he is moving his hand" [6, p. 260]. This is true because even if we don't know if this man is really writing or not, we know that if he is writing, then he surely is moving his hand, by definition. The two examples are reminiscent to al-Fārābī's kind of conditional ($2b_2$) above, which expresses an analytic implication, true by definition.

It seems then that the truth or the falsity of a conditional are determined on the basis of the meanings of its components, as all the examples show. Thus an implication, in Avicenna's frame, is not a material conditional. It is then intensional for it is based on the meanings of its elements.

In addition, Avicenna uses quantifications to express the several kinds of implications. These quantified implications are the following:

Ac: Whenever (*kullamā*) A is B then H is Z [6, p. 265] [whenever = always, if, then]

Ec: Never (= *laysa al battata*) (if A is B then H is Z) [6, p. 280]

Ic: Maybe (*qad yakūn*) if every A is B then every H is Z [6, p. 278]

Oc: Maybe not (if… then …) (*qad lā yaqūn*) [7, p. 235]

These quantifiers have been interpreted in two ways by different authors. Nicholas Rescher [13] says that they range over times, while Zia Movahed [12] says that they range over situations. In the first case, one can interpret them as temporal implications; in the second case, they would be more like modal implications, for the situations are not necessarily temporal, since they could also account for conceptual, for instance, mathematical cases. The temporal interpretation seems too weak to account for the conceptual and analytic link between the antecedent and the consequent; so the interpretation in terms of situations seems better because it is more general and more in accordance with the examples provided by Avicenna. However, one could not credit Avicenna with a theory similar to Possible Worlds Semantics, which is too sophisticated and mathematically inspired to be endorsed by a medieval author, nor even with a frame comparable to Carnap's theory of "state descriptions", which is also too related to modern (logical) probability theory to be evoked by Avicenna. The only possible worlds that Avicenna explicitly talks about are future worlds, but he evokes them in his modal logic rather than his hypothetical one, in the context of his analysis of the concept of possibility [6, p. 141].

Now what do the expressions (1) 'whenever if...then', (2) 'maybe if...then', (3) 'not whenever if...then' and (4) 'never if...then' really mean in Avicenna's frame?

(1) seems to express the *real* implication, i.e. the relation of 'following from' or of *entailment*, which is necessary and warrants the deductibility of the consequent. It is used in a universal affirmative proposition, that is, $\mathbf{A}_C$; (3) is $\mathbf{A}_C$'s contradictory and corresponds to $\mathbf{O}_C$, which contains a conjunction. For although what is literally said is 'if...then', what is *implicitly meant* is 'and' or something close to it. As to (2) and (4), they are also contradictory, for (4) is used in the universal negative proposition $\mathbf{E}_C$, while (2) is used in its contradictory $\mathbf{I}_C$, and expresses rather a conjunction.

These interpretations are clearly expressed in the following formalizations (where 's' stands for 'situation(s)'):

$$\mathbf{A}c = (\forall s)(Ps \supset Qs) \ / \ {\sim}\mathbf{A}c = {\sim}(\forall s)(Ps \supset Qs) = (\exists s)(Ps \wedge {\sim}Qs) = \mathbf{O}c$$

$$\mathbf{E}c = (\forall s)(Ps \supset {\sim}Qs) = {\sim}(\exists s)(Ps \wedge Qs) = {\sim}\mathbf{I}c, \text{ so } \mathbf{I}c = (\exists s)(Ps \wedge Qs)$$

Thus it seems that 'if ... then' in $\mathbf{I}_C$ and $\mathbf{O}_C$ is close to 'and', which seems to be its *implicit* or *presupposed* meaning, that is, its *pragmatic* meaning; while in $\mathbf{A}_C$ and $\mathbf{E}_C$, 'if...then' really expresses the relation of following from or of entailment, which is necessary, intensional and sometimes causal. The quantification on situations gives a modal connotation to all these propositions, for "(∃s)" means "for some situation", while "(∀s)" means "for all situations". So $\mathbf{I}_C$ would be interpreted as follows: "There is one (or more) situation(s) where P is the case and Q is the case", while $\mathbf{A}_C$ would be interpreted as saying: "In all situations, if P is the case, then Q is the case". These interpretations account for the *weak* and the *strong* meanings of the conditional. As to the *medium* meaning, Avicenna does not render it in a formal way and he does not seem to give it much importance, although he evokes it in his informal discussion of the conditional. Nevertheless, if one wants to express it in a more precise way, one could perhaps appeal to the concept of *imprecise probabilities* (expressed by the word "likely") and interpret it as saying something like: "In most situations, if P is the case, then Q is the case". But Avicenna himself never used the word "most" in his quantifications.

Now what is presupposed in the meanings of the universal propositions? The answer lies in the hypothetical syllogistic constructed by Avicenna with the four propositions above, which duplicates the categorical syllogistic, for we can find in the hypothetical syllogistic the counterparts of all valid categorical syllogisms.

For instance, the hypothetical *Barbara* is expressed thus:

- Whenever A is B, then C is D (= $\mathbf{A}_C$)

- Whenever C is D, then H is Z (= $\mathbf{A}_C$)

- Therefore whenever A is B then H is Z ($\mathbf{A}_C$) [6, p. 296]

Among these syllogisms, we find the hypothetical analogue of *Darapti*, for instance, which is expressed formally as follows:

$$[(\forall s)(Ps \supset Rs) \wedge (\forall s)(Ps \supset Qs)] \supset (\exists s)(Qs \wedge Rs).$$

As it stands, however, *Darapti* should not be valid, because the main conjunction may be true when the two premises are true, and the premises, which contain conditionals, might be true when their two propositions are false, in which case the conclusion which contains a conjunction, would be false. So why does Avicenna hold this mood valid, then? This is so because he presupposes that the antecedent of the two premises (of the form $\mathbf{A}_C$) must be true too. For when these antecedents are true, the consequents in the two universal premises of *Darapti* will be true; consequently the conclusion will be true too. This presupposition appears in the following quotation:

"When we say: 'If A is B, then H is Z', we assume from this (*nūjibu min hādhā*) that at any time where 'A is B' is the case and when A is B then H is Z, as if the fact that H is Z follows the fact that A is B, in so far as in effect A is B (*min ḥaythu hūwa kā'inun A [hūwa] B)*" [6, p. 263. 8–9]

This means that the real implication in $\mathbf{A}_C$ *presupposes* that the antecedent of the universal affirmative is true. Only in that case, the hypothetical *Darapti* could be valid; otherwise it is not. The first premise of *Darapti* above would then be expressed by the following conjunction '$(\exists s)Ps \wedge (\forall s)(Ps \supset Rs)$], which stipulates that the antecedent is true [9, p. 194].

What about $\mathbf{E}_C$? Does it require the same presupposition?

Given the syllogisms containing $\mathbf{E}_C$, no presupposition of that kind is required in $\mathbf{E}_C$. For instance, the hypothetical *Felapton*, which contains $\mathbf{E}_C$ and $\mathbf{A}_C$ is valid only when we presuppose that the antecedent of $\mathbf{A}_C$ is true. Nothing else is required.

Felapton is formalized as follows:

$$[\sim(\exists s)(Ps \wedge Rs) \wedge (\forall s)(Ps \supset Qs)] \supset (\exists s)(Qs \wedge \sim Rs).$$

Given the presupposition related to $\mathbf{A}_C$, 'Ps' is true. In that case, Qs must be true, in order for the whole implication to be true; so ~Rs must be true too, in order for the conclusion to be true; consequently Rs is false, being the contradictory of ~Rs. Since Rs is false, $\mathbf{E}_C$, here, is true without any further requirement.

Does this mean that the meanings of the implications in $\mathbf{A}_C$ and $\mathbf{E}_C$ are different? We might say that $\mathbf{A}_C$ and $\mathbf{E}_C$ have the same *semantic* meaning ('q *necessarily follows* from p' in the first case, and 'not q *necessarily follows* from p' in the second one) but not the same *pragmatic* meaning, for they do not require the same presuppositions.

4. Conclusion

Implications are intensional in both frames, for they depend on the meanings of the elements involved. However, the two authors analyse the different kinds of implication in different ways. Al-Fārābī distinguishes between an accidental implication, a regular and frequent one and a necessary one, which is itself sub-divided into two kinds: an incomplete one and a complete one. The semantic as well as the pragmatic meanings of all these kinds are different for they are not true in the same conditions and do not carry the same presuppositions. All of them seem to be intensional for their elements in all cases are related semantically and / or causally, but while the accidental implication exemplified by (1) above involves mainly a succession in time which may occur only once, the regular implication illustrated by (2a) involves a frequent succession in time and a *probable* causal or at least natural link between the antecedent and the consequent. As to necessary implications, they involve either a causal or a conceptual link between the antecedent and the consequent. When the implication is complete, the relation is convertible and its pragmatic meaning is an intensional and necessary biconditional. When it is not complete, the implication pragmatically expresses a necessary inclusion. In both cases, the relations have a modal connotation, but al-Fārābī does not use explicitly the word 'necessarily' to express them.

As to Avicenna, he does not admit the complete implication endorsed by al-Fārābī for formal and syntactical reasons, but he does distinguish between a weak, a medium and a strong implication which are expressed by different words in the natural language he is considering. His analysis of these different implications is more developed than that of al-Fārābī, although it is clearly influenced by it, for he enters into more details when analysing the causal links between the antecedent and the consequent in implications. According to him, the weak implication does not really involve a semantic relation between its elements, for it expresses what he calls '*ittifāq*', which has been translated as 'chance connection' or as 'agreement' by different authors, but seems rather close to the agreement with the facts if we consider Avicenna's explanations. The '*ittifāq*' is opposed to the real implication, called '*luzūm*', which is close to the notion of *entailment* or the relation of 'following from'. This last relation is intensional, necessary and universal, for it can be expressed by the word 'whenever'. But it is not truth-functional.

Avicenna expresses the implications used in his hypothetical logic by means of existential and universal quantifications. These quantifications give them modal connotations, for the quantifiers may range over situations. When formalized, the existentially quantified propositions express a conjunction, while the universally quantified ones express intensional entailments.

So the implicit and pragmatic meanings of the particular conditionals $\mathbf{I}_C$ and $\mathbf{O}_C$ seem close, since they both are expressible by conjunctions, while $\mathbf{A}_C$ and $\mathbf{E}_C$, although they carry the same semantic meaning, which involves a universal and necessary dependence relation, do not have the same pragmatic meaning, for $\mathbf{A}_C$ presupposes the truth of its antecedent. This presupposition is strongly suggested in Avicenna's hypothetical syllogistic, for it is required to validate some kinds of third figure hypothetical syllogisms. Such a presupposition is not required for $\mathbf{E}_C$, which is thus pragmatically different from $\mathbf{A}_C$.

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Implications and Limits of Sequences

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Abstract: This paper analyzes the problem of implication and attempts to characterize conditionals by a criterion of adequacy. A definition of implication based on the notion of limit of an infinite sequence is proposed.
Keywords: implications, conditionals, conjunctive limit, disjunctive limit

1. Introduction

Conditionals (or implications) are important operators especially because they are connected, in many cases, with consequence relations, as displayed in the famous deduction theorem. A conditional, at least in classical propositional logic, is a binary truth-function usually defined on a set {T,F} of truth-values where T refers to *truth* and F refers to *falsity*. More precisely, semantically speaking, a truth-function $f:\{T,F\}\times\{T,F\}\rightarrow\{T,F\}$ is a classical implication if, and only if,

$$f(T,T)=T;\ f(T,F)=F;\ f(F,T)=T,\ f(F,F)=T.$$

This connective (i.e. material implication) is a bivalent binary truth-functional operator. But these properties hardly define what an implication is because there are connectives which deserve to be called implications and which cannot be explicitly defined using a bivalent semantical characterization, as the case of some implications in many-valued logics. Moreover, there are also non-truth-functional implications as the case of strict and discussive implications. Still, one can find implications with a behavior not based on truth-values, as the case of intuitionistic and linear implication. Therefore, it seems essential to find a criterion of adequacy able to determine when a given binary operator is an implication. This criterion, of course, is not able to determine univocally and describe precisely what an implication is, but it can, at least, guide us in discovering some clues about what implications really are, and which properties these connectives are supposed to have.

A study concerning the nature of implication is rather relevant, assuming that this operator is connected to the very essence of logic. Besides being a crucial logical connective, conditionals are often found in formulations of central notions in science and mathematics. By the way, some theorems are stated in an implicative form: the fundamental theorems of arithmetic, algebra and calculus could all be expressed by conditionals.

In this paper, we give two contributions: first, we use the notion of *inferential systems* to suggest a criterion of adequacy for conditionals. Then, we use limits of infinite sequences, a simple notion from mathematical analysis, to define implication.

2. Inferential Systems and Conditionals

The word *implication* is understood as something which refers to linguistic entities. Informally: implication is an operation in an algebra of propositions. More specifically, if $\mathbf{P}$ is a set whose elements are called *sentences*, then an *implication* is something like a function $f:\mathbf{P}\times\mathbf{P}\rightarrow\mathbf{P}$ such that for each pair $\langle p,q\rangle$ of elements of $\mathbf{P}\times\mathbf{P}$ we get an element $f\langle p,q\rangle$ of $\mathbf{P}$ denoted by $p\rightarrow q$. There is, however, an ambiguity that should be explained. We use the word *implication* to refer to the function f as well as to elements of the image of f. In the first case, we say that f is the symbol of implication and in the second case $(p\rightarrow q)$ is the implication of the sentence q by p.

Assuming all above, a problem naturally arises: what are the conditions for a function of the type $f:\mathbf{P}\times\mathbf{P}\rightarrow\mathbf{P}$ be legitimately qualified as an implication? This is a question which is at the heart of what we call the *problem of implication*: how to legitimately characterize an implication? Indeed, this question is not an original one. Dunn and Hardegree ask in [3] the following version of the problem examined in this paper: “What are the (minimum) properties of a lattice operation, in virtue of which it is deemed an implication operation?” (p. 106). In a nutshell, given a complemented lattice, they answer this question providing the following requirements for an operation to be qualified as an *implication* ([3], pp. 105-109):

- Implication is a binary operation: they use a lattice, so implication is a partial order;
- (c1) (Truth) $p\leq q$ iff $p\rightarrow q=1$;
- (c2) (Modus Ponens) $p\wedge(p\rightarrow q)\leq q$;
- (c3) (Modus Tollens) $-q\wedge(p\rightarrow q)\leq -p$;
- (c4) (Refutation) $p\wedge -q\leq -(p\rightarrow q)$

Concerning these criteria, they state that: “Conditions (c1)-(c4) are collectively referred to as the *minimal implicative conditions*: every implication operation should satisfy all four conditions on any lattice with complementation.” ([3], p. 108). Let's call this the *algebraic* solution to the problem of implication. Although it is very elegant to define implication algebraically, the fact that complement is used commits the solution with extra entities which could be avoided in order to define implication. In our opinion, to use forms of negation to define properties an implication should satisfy is not essential.

The algebraic solution to the problem of implication is dependent of complements (i.e. negations). So, it is not pure. A pure solution is provided by Koslow who looks for conditions on implications in [4] without appealing to extra logical entities. Koslow's idea consists in adopting an implication structure as a pair $\langle \mathbf{S}, \Rightarrow \rangle$ such that $\mathbf{S}$ is not empty and the implication relation $\Rightarrow$ is defined as $\Rightarrow \subseteq \wp^{<\infty}(\mathbf{S}) \times \mathbf{S}$, where $\wp^{<\infty}(\mathbf{S})$ denotes finite subsets of $\mathbf{S}$, and it obeys some laws reproduced below ([4], p. 5):

- (Projection) $p_1,...,p_n \Rightarrow p_k$, for any $k=1,...,n$;
- (Simplification) If $p_1, p_1,...,p_n \Rightarrow q$, then $p_1,...,p_n \Rightarrow q$;
- (Permutation) If $p_1,...,p_n \Rightarrow q$, then $p_{f_1},...,p_{f_n} \Rightarrow q$, for any bijection f of $\{1,2,3,...,n\}$;
- (Cut) If $p_1,...,p_n \Rightarrow q$ and $q, q_1,...,q_m \Rightarrow r$, then $p_1,...,p_n, q_1,...,q_m \Rightarrow r$.

Koslow uses, in fact, six properties, but he argues that two other conditions, (Reflexivity) and (Dilution), can be derived from the properties above. He makes no use of any connective to define an implication relation. Structuralist logic proposed by Koslow assumes that "...implication is central to logic, and that the logical operators, as well as modal operators generally, can be characterized as such, by the way they interact with respect to implication."([5], p.167). Many logical structures such as the Tarskian consequence operator can be viewed as particular cases of implication structures (see [4], p.43). The structuralist theory of logic is so general in its understanding of what a logic is that it clearly anticipates what is currently known as *universal logic* (a concept developed in [2]). The notion of *implication structure* is used to define *hypotheticals* (i.e. Koslow's terminology for conditionals) (see [4], p.77-78). These are obtained from an implication structure if, and only if, two conditions are respected: the first one is *modus ponens* and the second one is the deduction theorem. This is Koslow's criterion of adequacy for conditionals. Let's call this the *structural* solution to the problem of implication.

Koslow's solution sounds better than that of Dunn and Hardegree, given that it is negation-free. Despite this fact, Koslow's answer to the problem of implication is committed to many general abstract inference rules, as he imposes many properties on implication relations. We think that something simpler can be obtained. Moreover, note that such a problem of implication could be approached at least in two different ways. From one side, we could search for a descriptive solution to the problem indicating the way in which implication is, in fact, characterized in language, and this seems to be the approach of Dunn and Hardegree, and also that of Koslow. They both try to describe what implications really are (descriptive solutions are, therefore, always open to refutations, since one can show they are false). From the other side, we could search for a normative solution showing how an implication should be characterized in a language. This kind of normative solution can only be properly judged by its utility, and not for its correctness. In a different way, we look for a normative solution, as we believe descriptive solutions have limits which cannot be transposed.

In order to present a proposal to the problem of implication we consider another concept strongly related to that of implication. It is the general concept of *inferential system*. With the aim of proposing a legitimate use of the concept of implication, we have to understand how it behaves with respect to certain admissible inferences inside an inferential system. This concept can be introduced from a general viewpoint.

An *inferential system* is a pair $\langle \mathbf{P}, \vdash \rangle$ such that $\mathbf{P}$ is a set of sentences and $\vdash$ is a consequence relation defined in $\wp(\mathbf{P}) \times \mathbf{P}$. Elements of $\vdash$ are, therefore, pairs $\langle \Gamma, \varphi \rangle$ such that Γ is a subset of $\mathbf{P}$ while φ is an element of $\mathbf{P}$. To point out that $\langle \Gamma, \varphi \rangle$ is a member of $\vdash$ the notation $\Gamma \vdash \varphi$ is used, and we say that φ is a consequence of Γ in $\langle \Gamma, \varphi \rangle$. We proceed as in [1]. So, we do not need to specify any kind of property that $\vdash$ should satisfy. Then, we can now introduce a proposal showing how to characterize implications inside inferential systems. The adequacy criterion for implications suggested is the following: Let $\langle \mathbf{P}, | - \rangle$ be an inferential system. A function $f: \mathbf{P} \times \mathbf{P} \to \mathbf{P}$, $\langle p, q \rangle \mapsto f(p,q) = (p \to q)$ is an implication with respect to $\langle \mathbf{P}, | - \rangle$ if, and only if, the following conditions are satisfied, for all subsets Γ of $\mathbf{P}$ and sentences p and q:

- Rule of *modus ponens* (MP): if $\Gamma | - f(p,q)$ and $\Gamma | - p$, then $\Gamma | - q$;
- Rule of reflexivity (R): $\Gamma | - f(p,p)$.
- Deduction theorem (DT): $\Gamma, p | - q$ iff $\Gamma | - f(p,q)$

Note that the criteria above are sufficiently broad to capture cases in which $| -$ is defined semantically, syntactically or in any other specific way. An implication is *perfect* if, and only if, besides validating (MP) and (R), it satisfies (DT). A degenerated implication is any implication which does not satisfy all conditions above. And, for us, an ideal implication is a *perfect* implication. In classical logic, it is easy to check that there is an operator which is a perfect implication: material implication clearly satisfies both (MP), (R) and (DT). A few comments on these conditions are called for. (MP) sounds essential for implication because it deals with truth-preservation, be it semantical or syntactical, and (MP) alone cannot define implication. (R) assures that neither conjunctions nor disjunctions are implications, and this cannot be assured only by an operator satisfying (MP). The deduction theorem sounds important as it connects implication with the central notion of logic, i.e., logical consequence.

Our solution looks very similar to that proposed by Koslow, with some important distinctions: Implication structures are committed to many properties on the implication relation, while our inferential system does not have any property at all. So, inferential systems are much more general. And we also impose reflexivity as a condition, while Koslow's criterion uses only (MP) and (DT).

Before introducing the main contribution of this paper, that is, the definition of implication by means of limits of infinite sequences, let's note that there are in the literature, however, many sorts of conditionals. This is a remarkable list: material, strict, counterfactual, linear, discussive, a great variety of many-valued conditionals and so on. Are all these perfect implications according to the criterion to determine whether a binary relation is or not an implication? Certainly not. In the same way that there are negations which are not truth-functional and which behavior is totally different from classical negation, we still call such operators negations. It would be a dogmatic attitude just to say that they are not negations because they do not behave like classical negation. The same situation happens here. Some implications are not like classical implication, and they do violate our criterion of adequacy, but they still deserve to be called implications, although they are not perfect implications.

3. Implications as Limits of Sequences

The next step we intend to develop is to get a notion of implication based on the notion of limit as it is used in mathematical analysis. Consider an inferential system $\langle \mathbf{P}, | - \rangle$ such that $\mathbf{P}$ is a set of formulas and $| -$ is a consequence relation. Be

$$(p_i)_{i \in \mathrm{N}} = p_0, p_1, \ldots, p_i, \ldots$$

a sequence of elements of $\mathbf{P}$. So we propose the following definitions:

Definition 3.1 (*Conjunctive limit*) *We say that* p *is the conjunctive limit of* $(p_i)_{i \in \mathrm{N}}$

$$\overset{c}{\lim_{i\to\infty}} p_i = p$$

if, and only if, $p \mid\!- p_i$ *, for all* $i \in \mathrm{N}$.

Definition 3.2 (*Disjunctive limit*) *We say that* p *is the disjunctive limit of* $(p_i)_{i\in\mathrm{N}}$

$$\overset{d}{\lim_{i\to\infty}} p_i = p$$

if, and only if, $p_i \mid\!- p$*, for each* $i \in \mathrm{N}$.

Consider now two sequences $(p_i)_{i\in\mathrm{N}}$ and $(q_i)_{i\in\mathrm{N}}$ such that conjunctive and disjunctive limits do exist respectively. Assume that there are sequences $(p_i \wedge q_i)_{i\in\mathrm{N}}$ and $(p_i \vee q_i)_{i\in\mathrm{N}}$.

Theorem 3.3 *The following holds:*

$$\overset{c}{\lim_{i\to\infty}} (p_i \wedge q_i) = \overset{c}{\lim_{i\to\infty}} p_i \wedge \overset{c}{\lim_{i\to\infty}} q_i$$

$$\overset{d}{\lim_{i\to\infty}} (p_i \vee q_i) = \overset{d}{\lim_{i\to\infty}} p_i \vee \overset{d}{\lim_{i\to\infty}} q_i$$

The proof follows from the definitions of conjunctive and disjunctive limits. It is straightforward to show that De Morgan's laws and classical material implication can be defined in this framework. To see this, we define the *negation* of a sequence $(p_i)_{i\in\mathrm{N}}$, in symbols, $\neg(p_i)_{i\in\mathrm{N}}$ as the sequence given by $(\neg p_i)_{i\in\mathrm{N}}$. Then, if the negation is the classical one, it is easy to see that:

Theorem 3.4 *The following holds:*

$$\overset{c}{\lim_{i\to\infty}} (\neg p_i) = \neg \overset{d}{\lim_{i\to\infty}} p_i$$

$$\overset{d}{\lim_{i\to\infty}} (\neg p_i) = \neg \overset{c}{\lim_{i\to\infty}} p_i$$

With negations at hands, it is easy to see how to define material implication. However, we do not want to use negation. So, now, in order to define what an implication is (without using classical negation), consider two propositions p and q.

Definition 3.5 *We say that* p *implies* q *(i.e.* $p \to q$*), if* q *is the disjunctive limit of a sequence in which* q *occurs. Therefore,* $p \to q$ *if, and only if, there exists a sequence* $(q_n)_{n\in\mathrm{N}}$ *such that*

$$\overset{d}{\lim_{n\to\infty}} (q_n) = q$$

and $p = q_i$, *for some* $i \in \mathbb{N}$.

This implication satisfies the three conditions (MP), (R) and (DT) exposed above. We prove that it satisfies (DT):

Theorem 3.6 $p \mid\!\!- q$ *if, and only if,* $p \rightarrow q$.

Proof. Assume first that $p \mid\!\!- q$. Take the sequence $(q_n)_{n\in\mathbb{N}}$ with $q_n = q$ for all $n \in \mathbb{N}$. In this case,

$$\overset{d}{\lim_{n\to\infty}}(q_n) = q$$

and, thus, $p \rightarrow q$. Now, assume that $p \rightarrow q$. Then, there exists a sequence $(q_n)_{n\in\mathbb{N}}$ with

$$\overset{d}{\lim_{n\to\infty}}(q_n) = q$$

and $q_i = p$, for some $i \in \mathbb{N}$. Thus, $q_i \mid\!\!- q$, i.e, $p \mid\!\!- q$. Q.E.D.

The original notions of conjunctive and disjunctive limits help us to define what an implication is from a general perspective, without using any other connective, in the same spirit of Koslow's Structuralist Logic.

4. Conclusion

We saw that the algebraic solution to the problem of implication has unnecessary commitments as it uses the notion of complementation (i.e. negation). We also saw that the structural solution is not too general and, by the end, implication relation is a sort of consequence relation rather than a connective. It is possible, despite this fact, to define implication as a connective and Koslow imposes only two conditions on it: (MP) and (DT). So, according to our viewpoint, hypotheticals are degenerated conditionals. These solutions are descriptive and they try to answer the question of what an implication really is.

As we have seen, our normative criterion of adequacy for conditionals sounds efficient as it is captures in a general and free mode some relevant properties implications should satisfy. As we believe, these are the main features for classifying an implication. From one side, a descriptive criterion is very difficult to provide, as it should be a criterion able to describe precisely all kinds of *conditionals*. On the other side, a normative criterion is feasible and it plays a pragmatic role.

By the end, we undertook an attempt to model logical connectives, and especially implication, using the notion of limits of infinite sequences. This is the main contribution here, and it is based on a connection between mathematical analysis and logic.

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Assertions and Conditionals: A Historical and Pragmatic Stance

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Abstract: The assertion candidate expresses a potential logical-linguistic object that can be asserted. It differs from both the act and the product of assertion; it needs not to be actually asserted and differs from the assertion made. We investigate the medieval origins of this notion, which are almost neglected in contemporary logic. Our historical analysis suggests an interpretation of the assertion candidate within the system of logic for pragmatics.
Keywords: assertion candidate, assertion sign, judgement, illocutionary force, logic for pragmatics.

1. Introduction

The theoretical nature of the notions of judgement and assertion is a key philosophical issue. A judgement is usually assumed to be the internal counterpart of an act of assertion [13] and this explains how the linguistic act of asserting is connected to the epistemic notion of judgement. In the present paper, an analysis of the concepts of judgement and assertion is provided by considering some aspects of both medieval and modern logic. Notably, we will focus on the formal system of illocutionary logic named *Logic for Pragmatics* (LP) [11], in which an assertion is justified by the existence of an *intuitive* but *correct* proof providing conclusive evidence for the truth of a proposition. Namely, proofs in LP are considered to be *factive*.

The notion of "assertion candidate", i.e. the specific linguistic entity that *can* be asserted, has been ruled out in almost all contemporary logical systems, even if it has a long tradition in the history of logic. One of the few contemporary examples of assertion (judgement) candidate in contemporary logic is offered by the intuitionistic type theory (see [23], as observed in [40]). Our objective is to provide an interpretation for the notion of assertion candidate in LP, without introducing a specific term for it in the language of LP.

According to van der Schaar [40], the assertion candidate:

1. is different from both act and product of assertion;
2. needs not be actually asserted; it is what *can* be asserted;
3. differs from the assertion made in that it has no [assertive] force;
4. when it is expressed by a sentence *S*, it has to be explained in terms of the condition under which one is entitled to assert that *S*.

Our interpretation aims at handling the notion of assertion candidate from a pragmatic perspective in accordance with the aforementioned conditions. Moreover, a critical analysis of the medieval developments of the notion of assertion candidate will be provided.

Section 1 traces the history of the notion of assertion candidate (generally called *enuntiabile*) and some developments on language from a medieval perspective. Section 2 presents a thorough analysis of the logical system LP. Section 3 points out Russell's "embedding problem" concerning the nature of assertability and inference along with a pragmatic treatment of such problem. Section 4 outlines how to properly interpret the notion of assertion candidate in LP. Finally, Section 5 shows that certain kinds of propositions, e.g. the (empirical) undecidable ones, cannot be assumed as assertion candidates in LP. It will be pointed out that the assertion candidate is interpreted in LP as a *description* of the illocutionary act of conjecture, which is associated with certain conditions of assertability in an inferential framework.

2. Medieval Background

This section provides an outline of the medieval development of the notion of proposition. It is essential for our discussion to specify the way in which the medieval and modern philosophers understand the term "proposition" (*propositio*), in order to avoid any confusion. Ever since Russell, this term has taken the form of a *that-clause*. According to the technical use of this term in philosophical language, a proposition is not a sentence, but its cognitive content (namely, what it expressed), e.g. "Socrates amat" and "Socrates loves" express the same thought for two different grammatical sentences. In the pre-Fregean tradition, as pointed out by van der Schaar [41], propositions had the form of declarative sentences. However, in the present paper, every time we make reference to a particular historical period, we will clarify the sense in which the term proposition and other related notions are conveyed by the authors.

For medieval grammarians and logicians the term "proposition" (*propositio* or *enuntiatio*) is synonymous with speech, which can be written, spoken and mental [25]. It is not a simple issue to unravel the relation between conventional meaning of written and spoken propositions and their mental counterparts. Such an intricacy stems from the complexity of Aristotle's sentences in *De Interpretatione* (16*a* 3-8) as well as from the unanimous (but misleading) interpretation of medieval commentators.[1] The common interpretation was that a written proposition (or its mental image) conventionally signified the corresponding spoken proposition, and in turn, a spoken proposition (or its mental image) signified the corresponding mental proposition. Thus, the written and spoken propositions accomplish their signifying function in subordination to the mental acts of apprehending and judging, always occurring before them.

An interesting position in this respect is that of Abelard [1], who emphasizes at the beginning of his commentary on Boethius' *De Interpretatione* [5] that the main object of investigation is the *propositio*, understood as a complex but unitary entity by which the name and the verb are to be conceived as components.[2] This aspect is closely connected to the distinction made by Abelard between the force and the content of an expression, which has been supported in modern logic by Frege, Russell, Geach, Searle and Vanderveken. According to Abelard, the same propositional content can be expressed with different force in different contexts: the statement that Socrates runs can be expressed by an assertion such as "Socrates runs", by a question as "Does Socrates run?", by an exclamation as "Socrates runs!", and so forth. Notably, Abelard distinguishes the assertive force of a sentence from its propositional content, a distinction that allows him to emphasize how the single components of a conditional statement are not asserted, but only the whole conditional is asserted. The statement "If Socrates is in a cell, then Socrates is a prisoner" does not assert that Socrates is in a cell and does not assert that Socrates is a prisoner, for neither the antecedent nor the consequent of an assertion of a whole conditional are asserted in this case. A pragmatic analysis of the "embedding problem" regarding the propositional contents involved in the assertion of a conditional will be provided in Section 3.

In the *Summa Logicae* (I, 1) [28] Ockham takes up the tripartite distinction of proposition,[3] pointing out that written and spoken propositions have the same meaning as the mental ones: the first two are conventional signs, while the latter is composed of natural signs. Nonetheless, he also partially restored the Aristotelian hierarchy of types of propositions since he considered the written proposition as a secondary conventional sign compared to the spoken one, and the spoken proposition as secondary and subordinate to the mental one. As a matter of fact, separate treatments of the semantics of terms and the semantics of propositions are justified by the Aristotelian distinction between two levels of speech and thought (*Categories* 1*a* 16,2*a* 4*; De Interpretatione* 16*a* 10). This means that there is (i) the level of names and verbs along with the thoughts corresponding to them, which do not yet involve any combination, and to which, therefore, it is not possible to apply the notions of truth and falsity; and (ii) the level of expressions and thoughts formed by a kind of combination that has to cope with truth and falsity (i.e. *complexio*). A *propositio* is then defined as a combination of words used to make known what is either true or false (*oratio verum falsumve significans*)[21].

It is worth noting that from the twelfth century onwards the notion of *enuntiabile* or *dictum* is occasionally used for indicating that something is assertable. Generally, the notion of *dictum* in the medieval tradition is the statement that comes in the form of a dependent clause (accusative *plus* infinitive). It is only starting from certain twelfth-century treaties that it begins to be also called *significatum*, while the expression "*enuntiabile*" appears to be an attempt to translate into Latin the expression λεχτόν of the Stoics. In the treatise *Ars Burana*, the terms *dictum*, *enuntiabile* and *significatum* are, in fact, used as synonymic terms: an *enuntiabile* is what can be asserted. In the treatise *Ars Meliduna*, assertable entities – called *enuntiabilia* – are not defined as substances or qualities, but seem to be characterized by a particular mode of being: they are inaccessible to the senses and can be merely grasped by thought.

Many medieval scholars were aware of the distinction between a *complexio* in the sense of mere predication – that is, without any assertive (or different) linguistic force – and a *complexio,* which is, instead, accompanied by an act of judging or asserting something. For instance, in the Prologue to *Ordinatio* (q. I, 55) [27], Ockham tries to clarify this doctrine and examines how the intellect can obtain knowledge through propositions.[4] Ockham distinguishes, in fact, three conditions leading to a judgement:

1. the *incomplex* cognition of the terms of a proposition (e.g., “Plato” and “man”);
2. the complex cognition of terms as a whole proposition (e.g., “Plato is a man”);
3. the act of assent or dissent towards the proposition, judged as either true or false (e.g., “Plato is a man is true”).

Ockham also pointed out that *incomplex* cognition (of the terms of a proposition) and complex cognition (a simple act that associates a predicative-term to a subject) jointly constitute *apprehensions*. A mental proposition without an act of assent or dissent is called *notitia apprehensiva*, whereas a mental proposition containing an act of judgement (of assent or dissent) is considered a ‘description’ of the world and is called *notitia adhesiva* or *iudicativa*.[5] There are two types of apprehension: while the first type involves the formation of mental propositions through an act aimed at apprehending things in a compound or divided way, the second involves whole mental propositions through an act of conceiving. The act of assent or dissent is directed towards the *incomplex* (terms) or the *complex* (whole proposition), namely towards what can be true or false. From this point of view, it follows that every act of assent or dissent presupposes apprehension according to the complex as well as to the *incomplex*. Referring to the aforementioned condition 3, Nuchelmans [26] clarifies that Ockham distinguishes between two kinds of assent or dissent:

1. The first one “is the act of acknowledgement that something is the case without any reflective apprehension of the mental proposition” [26, p. 79]), leading to intuitive knowledge of something (e.g. the *acknowledgement* of the terms “Plato” and “man”).
2. The second one is an act of judgement that the mental proposition is true (e.g. the *judgement* that “Plato is a man is true”).

Only the second act can be properly called “judgement”. This last distinction is treated in the *Quodlibetal Questions* (see III, q, 6; IV, q, 6; VII, q, 6) and also, partially, in *Reportatio* (see *Questions* 15 and 25). Ockham states that this manner of conceiving the nature of judgments is also evident from the sixth book of Aristotle’s *Ethics*, where he affirmed the existence of several *habitus* like understanding, knowledge, etc.

Ockham’s distinctions were generally accepted in late scholasticism. However, Jeronimo Pardo, in his *Medulla Dyalectices* [30],[6] pointed out that there must be an apprehensive cognition that needs to be different from the judicative one; in other words, a previous apprehensive cognition is required for the formation of any act of judgement.

Let us now consider Bricot’s notable position [6], which has been exposed in the *Quaestiones super totam logicam Aristotelis*. In the final part, containing the commentary on the *Posterior Analytics I* of Aristotle (*Dubitatur tertio*), Bricot wonders if and how *notitia adhesiva* may be distinguished from *notitia apprehensiva*. Right after, in *Arguitur primo*, he argues that *notitia adhesiva* presupposes *notitia apprehensiva* and is posterior to it;[7] then, in *Arguitur quarto*, it is stated that the act of judgement presupposes an act of apprehension. Moreover, in the same section, it is also reported that the *notitia adhesiva*, not the *apprehensiva*, is justified by means of a proof.[8] This perspective will be relevant for our treatment of assertion, since it fulfills (at least partially) the four conditions governing the assertion candidate. Moreover, Bricot’s perspective shows certain connections with our pragmatic treatment of assertions, which are, in our perspective, justified by means of proofs.

After Bricot some later authors, notably late-Scotists as well as representatives of the Thomistic school, distinguished a mere apprehensive proposition (that could be a *potential object* of *judgment* or *assertion*) from the proposition representing a judgment[9]. What has changed in the two aforementioned traditions is the name given to these types of proposition. Basically, there is no theoretical change in the two traditions but only a slight change in the terminology defining the types of propositions. A mere apprehensive proposition is defined by the representatives of the Scotistic tradition as *propositio mentalis obiectiva*, whilst the Thomists define it as *propositio enuntiativa*. As to the type of proposition expressing a judgment, the Scotists describe it as *propositio mentalis formalis*, while the Thomists regard it as *propositio iudicativa*.

3. Logic for Pragmatics: the Elements

The distinction between judgement and predication is essential for the pragmatic analysis of sentences, which was also suggested by Frege [16].[10] According to Frege, there is a clear distinction between thoughts and judgements. Namely, a thought has a truth value, while a judgement is the acknowledgment of the truth of a thought. An assertion is the external counterpart of a judgement and is expressed by mean of a specific sign. The assertion sign "$\vdash$" consists of two parts: the horizontal stroke "-", expressing that the content is judgeable and the vertical stroke "|", indicating that an assertion has been made. In Frege's system, no assertive sentence contains more than one pragmatic sign. Reichenbach observed that assertions are part of the pragmatic aspects of language and cannot be connected by means of truth-functional connectives.[11] He also pointed out that "assertion" is used in three different ways:

1) it denotes, first, the act of asserting
2) the result of this act, i.e., an expression of the form "$\vdash p$"
3) a statement which is asserted, i.e., a statement "p" occurring within an expression "$\vdash p$".

At any rate, there is no explicit explanation of what counts as a *potential "object" of assertion*, contrary to what occurred in medieval logic.

In LP, $\vdash p$ may be unjustified if there is no conclusive evidence confirming p. Therefore, p does not necessarily stand for an entity that *can* be part of a justified assertion when the possibility to get complete evidence (proof) is ruled out. Dalla Pozza and Garola, in their logical system named *Logic for Pragmatics* (LP) [11], provided a formal treatment of assertion, introducing pragmatic connectives. They suggested a pragmatic interpretation of intuitionistic propositional logic as a pragmatic logic for assertions, inspired by Austin's views on assertion (see [2]) and Searle and Vanderveken's theory of illocutionary logic [37]. According to the LP system, propositions can be either true or false, while the judgements expressed as assertions can be justified (J) or unjustified (U). Assertions in LP are purely logical entities, without any reference to the speaker's intentions or beliefs [11].

LP consists of two sets of formulas: radical and sentential. Interestingly, every sentential formula contains at least a radical formula as a proper sub-formula. Radical formulas are semantically interpreted by assigning them a (classical) truth value, while sentential formulas are pragmatically evaluated by assigning them a justification value (J, U) defined in terms of the intuitive notion of proof. The pragmatic language LP is the following:

Descriptive signs: propositional letters: $p, \gamma_1, \gamma_2, \ldots$
Logical signs for radical formulas: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.
Logical signs for sentential formulas: the sign of pragmatic illocutionary force
("$\vdash$" assertion);
the pragmatic connectives: $\sim$ pragmatic negation, $\cap$ pragmatic conjunction, $\cup$ pragmatic disjunction, $\supset$ pragmatic implication, $\equiv$ pragmatic equivalence.

Formation Rules (FRs)

Radical formulas (*rfs*) are recursively defined by the following FRs:

FR1 (atomic formulas): every propositional letter is a *rf*

FR2 (molecular formulas):

(i) Let γ be a *rf*, then $\neg\gamma$ is a *rf*
(ii) Let γ_1 and γ_2 be *rfs*, then $\neg\gamma$, $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$, $\gamma_1 \rightarrow \gamma_2$, $\gamma_1 \leftrightarrow \gamma_2$ are *rfs*

Sentential formulas (*sfs*) are recursively defined by the following FRs:

FR3 (elementary formulas): Let γ be a *rf*, then $\vdash\gamma$ is a *sf*

FR4 (complex formulas):

(i) Let δ be a *sf*, then $\sim\delta$ is a *sf*
(ii) Let δ_1 and δ_2 be *sfs*, then $\sim\delta$, $\delta_1 \cap \delta_2$, $\delta_1 \cup \delta_2$, $\delta_1 \supset \delta_2$, $\delta_1 \equiv \delta_2$ are *sfs*.

Thus, every radical formula of LP has a truth value. And every sentential formula has a justification value that is defined in terms of the intuitive notion of proof and depends on the truth value of its radical sub-formulas. The semantics of these radical formulas is the same as for classical logic: it provides the interpretation for the radical formulas, by assigning them a truth value and for propositional connectives as truth functions in the standard way. LP has both a classical fragment (CLP) and an intuitionistic fragment (ILP). CLP is the fragment of LP without pragmatic connectives.

Axioms for CLP are the following:

$$A_i \ \vdash (\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_1))$$

$$A_{ii} \ \vdash ((\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_3)) \rightarrow ((\gamma_1 \rightarrow \gamma_2) \rightarrow (\gamma_1 \rightarrow \gamma_3)))$$

$$A_{iii} \ \vdash ((\neg\gamma_2 \rightarrow \neg\gamma_1) \rightarrow ((\neg\gamma_2 \rightarrow \gamma_1) \rightarrow \gamma_2))$$

Modus ponens rule for CLP is:

[MPP] if $\vdash\gamma_1$, $\vdash(\gamma_1 \rightarrow \gamma_2)$, then $\vdash\gamma_2$

The semantic rules are the commonly used classical Tarskian rules σ; thus, regulating the semantic interpretation of LP. Let γ_1, γ_2 be radical formulas and 0 = false and 1 = truth; then:

(i) $\sigma(\neg\gamma_1)=1$ iff $\sigma(\gamma_1)=0$
(ii) $\sigma(\gamma_1 \wedge \gamma_2)=1$ iff $\sigma(\gamma_1)=1$ and $\sigma(\gamma_2)=1$
(iii) $\sigma(\gamma_1 \vee \gamma_2)=1$ iff $\sigma(\gamma_1)=1$ or $\sigma(\gamma_2)=1$
(iv) $\sigma(\gamma_1 \rightarrow \gamma_2)=1$ iff $\sigma(\gamma_1)=0$ or $\sigma(\gamma_2)=1$

Pragmatic connectives have a meaning, which is explicated by the *BHK* (Brouwer, Heyting, Kolmogorov) intended interpretation of intuitionistic logical constants. The philosophical interpretations of intuitionistic logic are, in fact, provided by means of notions like assertion, construction, problem.

The illocutionary force of assertion plays an essential role in determining the *pragmatic* component of the meaning of an elementary expression, together with the *semantic* component, i.e. the meaning expressed by radical formulas.

A pragmatic interpretation of LP is an ordered pair $<\{J, U\}, \pi_\sigma>$, where $\{J, U\}$ is the set of justification values and π_σ is the function of pragmatic evaluation in accordance with the following justification rules:

Justification Rules: They regulate the pragmatic evaluation π_σ, specifying the justification-conditions for the assertive formulas in function of the σ-assignments of truth-values for their radical sub-formulas:

JR1 – Let γ be a radical formula. $\pi_\sigma(\vdash\gamma)= J$ iff a proof exists that γ is true, i.e. that σ assigns to γ the value «true». $\pi_\sigma(\vdash\gamma)= U$ iff no proof exists that γ is true.

JR2 – Let δ be an assertive formula. Then, $\pi_\sigma(\sim\delta)= J$ iff a proof exists that δ is unjustified. i.e., that $\pi_\sigma(\delta)= U$.

JR3 - Let δ_1 and δ_2 be assertive formulas. Then:

(i) $\pi_\sigma(\delta_1 \cap \delta_2)= J$ iff $\pi_\sigma(\delta_1)=J$ and $\pi_\sigma(\delta_2)=J$;
(ii) $\pi_\sigma(\delta_1 \cup \delta_2)= J$ iff $\pi_\sigma(\delta_1)=J$ or $\pi_\sigma(\delta_2)=J$;
(iii) $\pi_\sigma(\delta_1 \supset \delta_2)= J$ iff a proof exists that $\pi_\sigma(\delta_2)=J$ whenever $\pi_\sigma(\delta_1)=J$.

The *Soundness Criterion* (SC) is the following:

Let be γ a *rf*, then $\pi_\sigma(\vdash\gamma)= J$ implies that $\sigma(\gamma)=1$.

SC states that if an assertion is justified, then the content of assertion is true. It is evident from the justification rules that sentential formulas have an intuitionistic-like formal behaviour and can be translated into the modal system S4, where $\Box\gamma$ means "there is an (intuitive) proof (conclusive evidence) for γ". A formula δ is pragmatically valid or p.valid (respectively invalid or p.invalid) if for every Tarskian semantic interpretation σ and for every pragmatic function of justification π_σ it follows that $\pi_\sigma(\delta)=J$ (respectively $\pi_\sigma(\delta)=U$). Notice that the intuitionistic fragment of LP, ILP, is composed by complex formulas with atomic radicals [11]. The axioms of ILP are the following:
ILP Axioms:

A1. $\delta_1 \supset (\delta_2 \supset \delta_1)$
A2. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\delta_2 \supset \delta_3)) \supset (\delta_1 \supset \delta_3))$
A3. $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$
A4. $(\delta_1 \cap \delta_2) \supset \delta_1$; $(\delta_1 \cap \delta_2) \supset \delta_2$
A5. $\delta_1 \supset (\delta_1 \cup \delta_2)$; $\delta_2 \supset (\delta_1 \cup \delta_2)$
A6. $(\delta_1 \supset \delta_3) \supset ((\delta_2 \supset \delta_3) \supset ((\delta_1 \cup \delta_2) \supset \delta_3))$
A7. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\sim\delta_2)) \supset (\sim\delta_1))$
A8. $\delta_1 \supset ((\sim\delta_1) \supset \delta_2)$

Modus ponens rule for ILP is:

[MPP'] if δ_1, $\delta_1 \supset \delta_2$, then δ_2

where δ_1 and δ_2 contain atomic radicals.

It is worth noting that justification rules do not always allow the determination of the justification value of a complex sentential formula when all the justification values of its components are known. For instance, $\pi_\sigma(\delta)$=J implies π_σ ($\sim\delta$)=U but not *vice versa* and π_σ ($\sim\delta$)=J implies $\pi_\sigma(\delta)$=U but not *vice versa*. In addition, a formula δ is pragmatically valid or p.valid (respectively invalid or p. invalid) if for every π and σ, the formula $\pi_\sigma(\delta$=) J (respectively π_σ (δ) =U). Hence, no principle analogous to the truth-functionality principle for classical connectives holds for the pragmatic connectives in LP, since pragmatic connectives are governed by *partial* functions of justification.

The modal (semantic) projection ( )* of pragmatic assertions is provided by the following translation in the modal system S_4, which provides a modal description of the pragmatic and illocutionary acts of assertion[12]:

$(\vdash \gamma)^*$	$\Box\, \gamma$
$(\sim \delta)^*$	$\Box \neg\ (\delta)^*$
$(\delta_1 \cap\ \delta_2)^*$	$(\delta_1)^* \wedge (\delta_2)^*$
$(\delta_1 \cup \delta_2)^*$	$(\delta_1)^* \vee (\delta_2)^*$
$(\delta_1 \supset \delta_2)^*$	$\Box\, ((\delta_1)^* \rightarrow (\delta_2)^*)$

Notice that assertions, as acts, can be justified or unjustified, while the modal formulas are a *description* of assertions which can be true or false.

Classical and intuitionistic formulas are formally related by means of the following "bridge principles" connecting classical and pragmatic connectives [11]:

(a) $(\vdash \neg \gamma) \supset\ (\sim(\vdash \gamma))$
(b) $((\vdash\gamma_1) \cap (\vdash \gamma_2)) \equiv (\vdash(\gamma_1 \wedge \gamma_2))$
(c) $((\vdash\gamma_1) \cup (\vdash \gamma_2)) \supset (\vdash(\gamma_1 \vee \gamma_2))$
(d) $(\vdash(\gamma_1 \rightarrow \gamma_2)) \supset (\vdash\gamma_1 \supset \vdash \gamma_2)$

It is worth noting that (a) – (d) show the formal relations between classical truth-functional connectives and pragmatic connectives. (a) states that the assertion of a negated proposition entails the pragmatic negation of the assertion, (b) shows that the conjunction of assertions is equivalent to the assertion of the conjuncts, (c) states that a disjunction of assertions implies the assertion of the disjuncts, while (d) indicates that truth-conditional implication implies pragmatic implication.[13]

4. Russell's "Embedding Problem": the Nature of Proof and the Inferential Role of Assertions

The notion of assertion has a fundamental inferential role in logic. For instance, a variety of perspectives on proofs and assertions has been carried out in recent years in constructivism. Proofs can be viewed in a myriad of ways; namely, as objects or processes, logical or empirical, temporal or eternalist, mind-dependent or absolute[14]. Although proofs provide a complete justification for assertive judgments, from an antirealist perspective, proofs are assumed to epistemically constrain (or to be equivalent to) intuitionistic truths. On the other hand, antirealist viewpoints do not always properly distinguish the semantic notion of truth from the pragmatic *criterion*, which is geared towards creating a method that establishes the truth value of a proposition. Unlike antirealists, Dalla Pozza & Garola [11] hold that what can be actual or potential is the pragmatic notion of judgement (assertion), not the semantic notion of truth, which relies on an eternalist perspective.

An important philosophical and logical problem seems to be associated with the conditions of assertability (and provability) in an inferential framework. Russell [38], in fact, observed that there is something odd in the standard account of *Modus Ponens*, namely in the inference: p, $p \rightarrow q$; therefore q: "the proposition "p implies q" *asserts* an implication, though it does not *assert* p or q.[15] The p and the q entering into this proposition are not strictly the same as the p or the q, which are separate propositions" (see [38, p. 35])[16]. This problem is named "embedding problem" and has a variant in meta-ethics known as the Frege-Geach problem[17] [16], [17]. As stated before, the *Modus ponens* rule in CLP is represented by the following argument:

MPP:

$(i^*) \vdash p$
$(ii^*) \vdash (p \rightarrow q)$
$\therefore (iii^*) \vdash q$

while in ILP, the *modus ponens rule* is represented by this different argument:

MPP':

$(i) \vdash p$
$(ii) \vdash p \supset \vdash q$
$\therefore (iii) \vdash q$

Note that (ii*) implies (ii) (bridge principle (d)). It is not difficult to show that Russell's objection to MP does not hold in the intuitionistic fragment of LP [12]. Notice that such objection can be overcome even in (an extension of) CLP, by making use of the bridge principle (d). In fact, at any step of an inference of LP, a logical law can be introduced. If we introduce (d) as an additional premise, then it follows that:

$(1) \vdash p$	premise
$(2) \vdash (p \rightarrow q)$	premise
$(3) (\vdash (p \rightarrow q)) \supset (\vdash p \supset \vdash q)$	bridge principle (d)
$(4) \vdash p \supset \vdash q$	from (*2*), (*3*) and MPP
$(5) \vdash q$	from (*1*) and (*4*) and MPP'

Henceforth, Russell's objection to the "embedding problem" can be overcome in LP. There exists, indeed, a kind of priority in the intuitionistic approach in the theory of deduction over the classical one in LP since inferences occur between assertions (judgments) that have an intuitionistic-like formal behaviour, and not between classical propositions. Moreover, it is also remarkable that (4) is justified by the rule JR3.3 for the intuitionistic conditional when the antecedent is also justified. Thus, unjustified antecedents in a conditional cannot justify a conditional.

5. The Assertion Candidate and Logic for Pragmatics

This section tries to provide an answer to the following question: can the illocutionary force be unrestrictedly applied to propositions in order to have justified assertions? In our framework, we will point out that assertion candidates can be interpreted in a pragmatic framework as (semantic and modal) descriptions of conjectures. A conjecture may be converted into a justified assertion if the proof (conclusive evidence) for its content becomes available. The description of a conjecture can be *potentially* asserted even if they may be *not effectively* asserted[18]. In this respect, it is important to distinguish *intrinsically undecidable propositions* (IUPs) from *contingently undecided propositions* (CUPs)[19]. On the one hand, if propositions are intrinsically undecidable then they cannot be converted into assertions even in line of principle; therefore, they cannot work as assertion candidates since they always remain unjustified. On the other hand, contingent undecided proposition might be converted into proper assertions when new evidence is available. An example of intrinsically undecidable proposition is due to Pap [29, p. 37][20]: "everybody in the universe, including our measuring rods, is constantly expanding, the rate of expansion being exactly the same for all bodies."

Pap's sentence is clearly not verifiable, even if it has a truth condition, namely that we know how the world should be in order to make the sentence true. Hence, Pap's sentence cannot be asserted and justified in line of principle and cannot work as an assertion candidate, for instance, as the antecedent of a conditional in a *BHK* framework. On the other hand, there are contingently undecided propositions that may possibly be verified, for instance in the future, such that they can be converted into assertions or can remain forever unknown.

The possibility of assertion in our pragmatic framework shows some similarities with the illocutionary act of conjecturing. In fact, the assertive sign is translated and described as $\Box$ in the modal system S4, the act of hypothesizing with $\Diamond$, while the act of conjecture can be translated as $\Diamond\Box$ in S4 (see [3]).[21] The modal translation is very important for our proposal, since it may provide a semantic descriptive version of a pragmatic act, thus expressible in the radical part of a sentential formula. Genuine illocutionary operators cannot be, in fact, nested [16] but, of course, the semantic description of an illocutionary act can be part of a radical formula. The assertion candidate in this case may be, thus, interpreted as the (modal) radical part of a sentential formula as $\vdash\Diamond\Box\, p$.[22] The radical part of this formula states that there exists the possibility to prove (and, therefore, to assert) *p*.

Let us take a closer look at the pragmatic notion of conjecture. A hypothesis is justified if there exists the epistemic possibility (at least a *scintilla of evidence*) grounding the content [10], while a conjecture expresses the *possibility to assert* the content (being not intrinsically undecided). For instance, Goldbach's Conjecture can be viewed as a statement that is actually not proven, even if there is the possibility that it may be demonstrated in the future.[23] On the contrary, since it is impossible to prove an *intrinsically undecidable proposition* and its (classical) negation, then these conditions can be expressed in LP in this way:

$$\text{IUP:}\ \sim\vdash\gamma \cap \sim\vdash\neg\gamma$$

The first conjunct of IUP can be translated in S4 as $\Box\neg\Box\,\gamma$, that is equivalent to $\neg\,\Diamond\Box\,\gamma$ (namely, the description of the fact that the conjecture γ is unjustified), while the second conjunct of IUP is translated in S4 as $\Box\neg\Box\neg\,\gamma$, that is equivalent to $\neg\,\Diamond\Box\neg\,\gamma$ meaning that the description of the fact that the conjecture $\neg\,\gamma$ is unjustified. Therefore, if γ is an intrinsically undecidable proposition, then γ as well as $\neg\,\gamma$ are not conjecturable. On the other hand, a contingently undecided proposition might be subsequently conjecturable once new evidence justifies the possibility of the assertion of the corresponding content. A contingently undecided proposition can be expressed in the meta-language of LP as:

$$\text{CUP: } \pi_\sigma(\vdash\neg\,\gamma) = \text{U and } \pi_\sigma(\vdash\gamma) = \text{U}$$

The modal translations of CUP are $\neg\,\Box\,\neg\,\gamma$ and $\neg\,\Box\,\gamma$, respectively $\Diamond\,\gamma$ and $\Diamond\neg\,\gamma$. This explains the contingency of such kind of propositions. Following our pragmatic interpretation, it seems that what can be asserted may be interpreted as the specific description of a conjecture, and, when applying the modal translation, also as a modal radical formula.

Indeed, our pragmatic interpretation of the assertion candidate fulfills van der Schaar's four conditions, namely:

(1) it is different from both the assertion act ($\vdash$) and the assertion product ($\vdash p$);
(2) it is what *can* be asserted in principle;
(3) it differs from the assertion made in that it has no [assertive] force;
(4) it is expressed in terms of the conditions of possibility under which one is entitled to assert a sentence.

Condition 1 pragmatically expresses the fact that it is necessary not to collapse the assertion act into the assertion product since the latter is the assertion sign *plus* the radical formula. The description of a conjecture expresses the (modal) radical formula that may be asserted. Condition 2 expresses that, from a pragmatic perspective, intrinsically undecidable propositions cannot work as assertion candidates since their possibility to be asserted is *a priori* ruled out. Condition 3 states that the assertion candidate, intended as the *description* of a conjecture, works as a specific radical formula, which does not have an assertive force, but that may be asserted and justified in case of conclusive evidence. Finally, condition 4 indicates the conjecturable nature of the content of an assertion, working as an assertion candidate.

6. Conclusion

The notion of assertion candidate has not been thoroughly analyzed in contemporary systems of logic. To meet this need, we have provided an interpretation of this notion within the framework of LP. Our pragmatic interpretation has been suggested by the medieval treatment of the notion of *enuntiabile*. The *enuntiabile*, in fact, is different from the act of assertion: it is not the result of such act but is rather what *can* be asserted.

Differently from other logical systems, in LP it is possible to distinguish the assertive force from its content and provide a way out of Russell's "embedding problem", which in itself is a major concern in philosophy of logic due to the justification of logical inference.

Our pragmatic treatment has clarified that assertion candidates can be interpreted as specific radical formulas that express a modal proposition with a possibility of being asserted. In LP, we have argued that assertion candidates can be viewed as the *description* of conjectures. However, we do not claim that assertion candidates are identical to conjectures, but rather that they can be interpreted in LP as the (modal) description of a conjecture.

Our analysis has also elucidated that IUPs cannot be asserted even in line of principle, whereas CUPs are those unjustified propositions that when integrated with new evidence do not reveal whether they may or may not become justified assertions. Lastly, both IUPs and CUPs have received a formal treatment in LP in order to isolate their logical and pragmatic features.

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Notes

1. In particular, see the Latin translation of Boethius' *De Interpretatione* [5];[18] (cfr. Boethius i 7.10 ff. in Meiser 1880): "Sunt ergo ea quae sunt in voce earum quae sunt in anima passionum notae, et ea quae scribuntur eorum quae sunt in voce. Et quemadmodum nec litterae omnibus eaedem, sic nec eaedem voces; quorum autem hae primorum notae, eaedem omnibus passiones animae sunt, et quorum hae similitudines, res etiam eaedem."
2. [1, pp. 307 (20-23) and 207].
3. Starting from the studies on obligationes, a distinction between different attitudes, namely *affirmo*, *nego*, and *dubito*, was quite common in Middle Age.
4. Being the medieval notion of proposition distinguished into mental, written and spoken, then the term "knowledge" is used here not as a substitute for the modern sense of propositional knowledge but in a broader sense.
5. "Est igitur prima distinctio ista quod inter actus intellectus sunt duo actus quorum unus est apprehensivus, et est respectu cuiuslibet quod potest terminare actum potentiae intellectivae, sive sit complexum sive incomplexum; quia apprehendimus non tantum incomplexa sed etiam propositiones et demonstrationes et impossibilia et necessaria et universaliter omnia quae respiciuntur a potentia intellectiva. Alius actus potest dici iudicativus, quo intellectus non tantum apprehendit obiectum sed etiam illi assentit vel dissentit. Et iste actus est tantum respectu complexi, quia nulli assentimus per intellectum nisi quod verum reputamus, nec dissentimus nisi quod falsum aestimamus. Et sic patet quod respectu complexi potest esse duplex actus, scilicet actus apprehensivus et actus iudicativus." […] "Prima conclusio praeambula est ista quod actus iudicativus respectu alicuius complexi praesupponit actum apprehensivum respectu eiusdem" (Ockham, *Scriptum in libros sententiarum*, *Prologue*, I, 1, O. Q.)
6. *Medulla Dyalectices* was written in 1505.
7. "notitia adhesiva presupponit notitia apprehensiva et est ea posterior, sed nihil est posterius semel positio. Igitur notitia adhesiva distinguitur ab apprehensiva" (Bricot, *Quaestiones super totam logicam Aristotelis*, Y, 5).
8. *Ibidem*: "Quarto sic notitia apprehensiva conclusionis non acquiritur per demonstrationem cum aliquum sit ante eam et tamen eius notitia adhesiva per eam acquiritur. Igitur notitia adhesiva distinguitur a notitia apprehensive."
9. See for Scotists, [39]; for Thomists in particular, see [19].

10. We focus on the Frege's distinction between predication and judgments. However, Descartes and Spinoza also previously discussed such distinction. These notions have played an important role also for the foundations of mathematics, especially at the origins of intuitionism [22].

11. See section 57, called "Logical terms in a pragmatic capacity", in [36]. However, we do not claim that there is direct influence of Frege's perspectives on Reichenbach as it happened to be with the Vienna Circle.

12. The modal translation of the pragmatic connectives in S4 is the standard one presented in [11].

13. Extensions and applications of LP to philosophical issues have been presented in [3], [4], [7], [9], [10].

14. The main antirealist perspectives regarding the nature of proofs and assertions are those of Dummett, Prawitz and Martin-Löf. Let us consider, first, Prawitz's picture on proof: "[A] mathematical sentence is true if there is a proof of it, in a tenseless or abstract sense of exists […]. That we can prove A is not to be understood as meaning that it is within our practical reach to prove A, but only that it is possible *in principle* to prove A" [33, pp. 153-154]. Dummett's replied that: "We can introduce such a notion [of eternalist proof] only by appeal to some platonistic conception of proofs as existing independently of our knowledge, that is, as abstract objects not brought into being by our thought" [14, pp. 258-9]. A more sophisticated version of antirealism is expressed by Martin-Löf: "That a proposition A is actually true means that A has been proved, that is, that a proof of A has been constructed, […] whereas to say that A is potentially true means that […] a proof of A can be constructed" [23, p. 142].

15. For instance, notice that in LP also the assertion of a disjunction is not equivalent with the disjunction of assertions. Russell's presentation of the embedding problem is done by considering Carroll's paradox of inference. For a recent work on it, see [24].

16. Quine [34, p. 12] states that "an affirmation of the form "if *p* then *q*" is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent", given the antecedent. In any case, some counterexamples to Quine's idea are easily conceivable.

17. We have pointed out that the distinction between assertive force and propositional content was already described in Abelard's writings.

18. One might wonder if the linguistic declarative mood *correlates* with or *encodes* with the type of force, notably the assertive force. By contrast, according to Recanati "declarative sentences do not correlate with any category of illocutionary force. They are illocutionarily neutral. A declarative sentence represents a state of affairs, […]; how the representation is interpreted (in illocutionary terms) is left to context. (Of course there is a blocking effect due to the competition with the other moods — those which do correlate with types of illocutionary force)" [35, p. 630]. Quite different is Dummett's perspective for which it is important to distinguish the mere *act* of assertion from the *point* with which the assertion is performed [13].The aim of assertion is truth, and assertions need to be publically recognized as such in order to be justified; nonetheless, assertions in natural language can be performed to convey different attitudes (points) of statement use.

19. Such distinction has been partially inspired by Dummett [15].

20. See also ([34], section II.1). A similar sentence is also mentioned in [20].

21. A stronger view on conjecture is expressed in [29], where a conjecture is expressed as $\Diamond\Box$ in the modal system S4.2.

22. For an extension of LP with modal formulas as radicals, see [8].

23. Of course, Goldbach's conjecture may also remain unsolved in the future or it may happen that a new theorem might state the impossibility to prove the content of the conjecture. An example of 'impossibility' theorem is given by the Abel–Ruffini theorem, stating that there is no general algebraic solution to polynomial equations of degree ≥ 5.

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Conditionals in Interaction

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Abstract: There are several issues with the standard approach to the relationship between conditionals and assertions, particularly when the antecedent of a conditional is (or may be) false. One prominent alternative is to say that conditionals do not express propositions, but rather make conditional assertions that may generate categorical assertions of the consequent in certain circumstances. However, this view has consequences that jar with standard interpretations of the relationship between proofs and assertion. Here, I analyse this relationship, and say that, on at least one understanding of proof, conditional assertions may reflect the dynamics of proving, which (sometimes) generate categorical assertions. In particular, when we think about the relationship between assertion and proof as rooted in a dialogical approach to both, the distinction between conditional and categorical assertions is quite natural.
Keywords: conditions, interaction, assertion, proof

1. Issues with Conditionals

There are well-worn issues with the way in which conditionals are supposed to be understood, particularly when they have false antecedents.

The standard treatment of conditionals tells us that $\alpha \rightarrow \beta$ is true whenever is false α.[1] But, in English, when we say that "α, therefore β", or "if α, β", if α were shown to be false, it does not seem correct to say that the conditionals are true. According to many critics [14], [32], [33], this (amongst other issues) suggests that asserting a conditional does not express a "conditional" proposition. That is to say, whatever is distinctive about uttering a conditional statement is not that it is an assertion that expresses a distinctive kind of propositional content.[2] Rather, such an assertion is to be thought of as a *conditional* assertion of the consequent, where the condition is the antecedent. This, conditional, assertion, then, is a distinctive speech act, as Stalnaker [38] points out, not a standard speech act (assertion) with distinctive content.

There are different ways of thinking about conditional assertions. For example, some proponents have thought that whenever the antecedent is false, the condition upon which the assertion of the consequent is made does not hold, and so no assertion is made. Quine puts this as follows:

An affirmation of the form If α then β is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. If on the other hand the other hand the antecedent turns out to have been false, our conditional affirmation is as if it had never been made [32, §3].

Then, as Humberstone [17] suggests, after Ramsey [33], to assert a conditional is not to be thought of as asserting a conditional proposition, but to make a conditional assertion of the consequent: 'If the latter condition is not satisfied (i.e., if the antecedent is false), then it is as though no assertion had been made. The parallel is with conditional bets, which are void in that no money changes hands unless the condition they are conditional upon obtains' (p. 938). This is also captured in the infamous statement by Ramsey:

> If two people are arguing "If p will q?", and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q [...] If either party believes not p for certain, the question ceases to mean anything to him except as a question about what follows from certain laws or hypotheses [33, p. 247].

Whilst this approach does seem to capture something correct about the treatment of conditionals with false antecedents, it is, nonetheless, tricky to explain how these conditional assertions that may fail to assert anything are supposed to interact with standard approaches to speech-acts, particularly in terms of how they interact with propositional content.[3] For example, take a conditional like the following: "If you press that switch, there will be an explosion". When stated by one agent to another, even if the antecedent does not hold (the hearer does not press the switch), it does not seem that nothing has been said, since the hearer has learned some sort of reason to think that, should the antecedent hold, then the consequence will be an explosion![4] This suggests that conditional assertions are not "empty", but that they are distinct from making a categorical assertion of the consequent. As Edgington [13] puts it: 'My hearer understands that if she presses [the switch], my assertion of the consequent has categorical force; and given that she takes me to be trustworthy and reliable, if it does acquire categorical force, it is much more likely to be true than false. So she too acquires reason to think that there will be an explosion if she presses it, and hence a reason not to press it' (p. 178). An additional, though less discussed issue for the approach, is that it does not cohere well with standard accounts of deduction. It is difficult to know whether an argument involving conditionals as premises or conclusion is valid. Take the following argument:

$$\dfrac{\begin{array}{c}[\alpha]\\ \beta\end{array}}{\alpha \to \beta}\,(\to -I)$$

where $[\alpha]$ indicates that α is an assumption. Now, say that α is false, then, the conditional assertion codified by the inference step from $[\alpha]$ to β expresses nothing, and, so presumably no inference is made. So, the movement from $[\alpha]$ to $\alpha \to \beta$ is disconnected, leaving a "gap" in the deduction.

What the standard approach to conditionals, and the conditional assertion approach share is the idea that categorical assertions are the "industry standard", as it were, with conditional assertions (if allowed) to be explained in terms of them. As a result, there is an "all or nothing" status awarded to assertions, so, inevitably, conditional assertions are thought to be empty if they do not become categorical. To the contrary, I here pursue the idea that conditional assertions may be better understood in terms of the *dynamics* of logical reasoning inside dialogical situations. This is to take conditional assertions as the norm, with categorical assertions generated through dialogical interaction, and in certain circumstances. That is to say, we can see conditional assertion in terms of the dynamic process of reasoning, with categorical assertion as the objects (sometimes) produced by that process.[5] To get there, I begin in § 2 by discussing the usual understanding of the relationship between proof and categorical assertion, and show that it rules out the conditional assertions approach altogether. In § 2.1, I draw attention to a divergent view of proofs, which emphasises the activity of proving, and suggests that there might be a place for conditional assertions alongside categorical assertions. I begin § 3 by drawing attention to an analagous approach to assertions in terms of the social dynamics of commitment, before, in § 3.1, showing that, by placing both assertions and proofs in a social context, a natural approach to conditional assertion emerges.

2. Deduction and Categorical Assertion

Let us consider natural deduction in intuitionistic form, primarily because this logic does not seem to force the categorical assertion view upon us in the same way as classical logic, where $\alpha \to \beta$ is equivalent with $\neg\alpha \vee \beta$. In fact, some of the ingredients of the conditional assertion view are already apparent in the Brouwer-Heyting-Kolmogorov (BHK) interpretation of a conditional: A proof for $\alpha \to \beta$ is a function f which, to each proof a of α provides a proof $f(a)$ of β. But, whilst this suggests that such an account is amenable to conditional assertion, this turns out not to be the case, at least according to the most prominent interpretation of the validity of proofs in natural deduction.

According to Prawitz and Dummett, validity is definitional of what a proof is, and its validity is relative to a formal entailment structure, so to ask whether or not a proof is valid is nonsensical.[6] Prawitz does, however, consider "closed" and "open" arguments, showing how to define validity for these such that a valid closed argument is equivalent with a proof. An open argument is just an argument that involves undischarged assumptions, or unbound variables. In contrast, a closed argument has no assumptions, and is valid just in case it is *either* canonical, so that it ends with an instance of an introduction rule (in natural deduction calculus), *or* it can be reduced to a canonical argument for the conclusion.[7] Then, according to Prawitz, it is also possible to say that a closed argument is valid iff it can be identified with a proof, and an open argument is valid if it can be reduced to a closed canonical argument:

Definition 1 *(Prawitz-Dummett definition of validity): An argument A is valid whenever:*
- A is closed and canonical;
- or A is closed and reduces to a canonical argument;
- or A is open and reduces to a closed canonical argument.

In other words, the emphasis is on closed and canonical arguments in a deductive system, which allows for the reduction of non-canonical to canonical arguments, and also ensures that for a canonical argument to be valid requires its immediate subproofs to be valid. The justification of open arguments relies upon this prior notion, by taking the open argument and replacing all open assumptions with closed proofs (or open variables with closed terms). So, in general, a proof of β under the assumption α is valid whenever it is possible to replace the assumption α with a (valid) closed proof of α.[8.]

This approach to the validity of arguments accords with an approach to assertions that privileges categorical assertions, and, in fact, is at odds with any appeal to conditional assertion. This is, in part due to the interpretation of proofs as providing the *objectively correct* conditions for an assertion:

> [...] in the general case, we have to consider as primary, in determining the content of an assertion, not the speaker's personal entitlement to make the assertion, but the condition for its objective correctness [9, p.120].

These objective correctness conditions for sentences involving logical constants are just formally derivable proofs (or valid arguments), which must be stable, objective, and timeless.[9] As such, this coheres with the, now, fairly dominant understanding of proofs as objective, a view found in Prawitz Prawitz [31] and Dummett [e.g. 12], which requires only that there exist an effectively decidable possible proof for a statement, and not an actual proof carried out by an agent.[10] Insofar as proofs are understood in objective terms, the *act* of proving is reduced to a kind of ratification, where a proof itself is unaffected by our interaction with it. In order for a sentence to be asserted, the kind of evidence that is required is a closed proof. Open proofs are assertible, only insofar as they can be reduced to closed proofs. That is, if we have an open proof of β from α, then β could not be asserted unless we have some evidence that α holds also. It is a small step to see that, on this view, the only kind of assertion of interest is categorical, since there is no "room for manoeuvre": either a proof exists or it doesn't, and whether or not a proof exists is what determines the correctness of assertions.

This is a general feature of this approach to the relationship between proof and assertion, but, unsurprisingly, this becomes most clear in the case of the conditional, by which it is possible transform an open proof into a closed proof. Take, for example, a standard derivation introducing a conditional:

$$\dfrac{\begin{matrix}[\alpha]\\ \beta\end{matrix}}{\alpha \rightarrow \beta}\,(\rightarrow -I)$$

Here, we have a proof of $\alpha \rightarrow \beta$, which, given a proof of β, no longer depends upon α as assumption. That is to say, it is closed and canonical, so it is a valid proof, by definition, which introduces the conditional statement, whether or not α holds.[11] By the connection between assertion conditions and proofs, we also know that, in this case, $\alpha \rightarrow \beta$ must be categorically assertible. This becomes even clearer when we consider negation, defined (intuitionistically) as $\neg\alpha =: \alpha \rightarrow \perp$ (where $\perp$ expresses a constantly false proposition), since, if we say that $\alpha \rightarrow \perp$ is assertible on condition that α holds, whilst, definitionally, neither $\perp$ nor α can not hold, then $\neg\alpha$ could never be asserted.[12] Conditional assertions are ruled out, by definition.

2.1 Conditional Dynamics

There is a different view of proofs, which coheres somewhat better with conditional assertions, and takes agents' proving activity to be central. This is to think of proofs as *acts*, rather than as objective entities. For example, the objective approach to proofs (as pointed out in [e.g. 23, p.84-5]) seems to require commitment to an objective realm of propositions, leading to an "inert platonism of proofs" [7]:

> [f]ar from being deduced or extracted by ourselves, the consequences of an hypothesis follow from it by themselves, or rather in virtue of the existence of certain objects that it is none of our responsibility to conceive, or to make up, but only to discern [...] By identifying proofs with sequences of formulas or, more generally, with objects that are independent from us, one almost unavoidably reduces the activity of justification to a scanning and control process that requires no cognitive or physical particular resource .... [6].

Instead, according to Dubucs and Marion [7] we should think of proofs as "acts", rather than "objects":

> We propose that one distinguishes between two different notions of proof, namely those of proof as 'object' and as 'act' . According to the first conception, a proof is something like an assemblage of strings of symbols satisfying such and such property. From the second, more dynamic, conception, a proof is a process whose result may be represented or described by means of linguistic symbols.

On this view, a proof is understood as an *act* that is undertaken by agents, allowing that there is some sort of *dynamics* to proofs insofar as they are carried out in time. That is, we may take the act of proving seriously insofar as logic is taken not just to deal with propositional and objective, but with the *actions* of reasons themselves. The process of "proving", then, is more like a process of reasoning that is not required to live up to objective correctness conditions on assertions, which may be thought of as *generating* proof-objects. The latter is something like an *actual* proof, which for a conditional, $\alpha \to \beta$, is just a function that maps *actual* proofs of the α into *actual* proofs of β. Such a function can not map on assumptions, according to this view, since then, the function could not be a map at all: 'as long as no proof of α is known, [the function] f has nothing to map. So we can still define f as the constant function which, once a proof π of α is known, maps every proof of α into the proof of β' [23, p. 91].[13] In this regard, of particular note is the distinction made by Martino and Usberti [23] between what we may call a *hypothetical* function f_H that would come into effect once we have a proof of the antecedent to hand, and the *actual* function f_A which maps the antecedent into the consequent when we have the proof of the antecedent. The latter is *actual* since the function is only then an actual map that has come into effect given that a proof of the antecedent of the conditional is available. Since the conditional is intended to "write-into" the object language the relationship defined by the turnstile, and by the functional definition of a proof given at the start of this section, we can generalise this distinction as follows. In general, a proof of $\alpha \mid\!- \beta$ is just a function that takes α, and maps it into β, hypothetical just in case the required evidence for α is not to hand, and actual just in case there is such evidence. As such, we now have a distinction between a hypothetical proof (in which we assume α), and a valid proof, in which case a proof of α is also given.[14] This, moreover, is a distinction *inside* the notion of a proof, which does not alter the overarching definition given at the beginning of this section.

The distinction between hypothetical and actual proofs upturns Prawitz's distinction between closed and open arguments, with hypothetical proofs (open arguments) the norm, which in a subset of cases generate actual proofs (closed arguments). We are also liberalising the notion of proof, since it is here conceived as an activity, which may or may not produce a actual, valid, proof. This is to take seriously the idea that logic is not simply a matter of consequence and the construction of valid proofs, but rather it (also) has to do with the act of proving, reasoning and the construction of judgements. As I say, above, the standard approach to the validity of a proof, stemming from Heyting [16] through Dummett [12], focuses on proof-objects. The central feature that these views share is that a proof of a formula α is a construction π such that π makes α true, and that knowing a proposition is to have a constructive proof of it. In distinction, we may follow Sundholm's [39] account of constructions, which argues for a process / product distinction *inside* constructions, regarded both as processes, or as those processes taken as objects. The idea, is that a proof-object is that which remains *posterior* to the completion of a proof-act, and the trace of a proof is what is written down as the recipe for how to construct that proof.[15] A proof is something that is carried out in time, which then may become an object only subsequently, and in this sense, we also follow Martin-Löf's [22] argument that: '[a] proof is, not an object, but an act [...], and the act is primarily the act as it is being performed, only secondarily, and irrevocably, does it become the act that has been performed'. The process of "proving", then, is more like a process of reasoning that is not required to live up to objective correctness conditions on assertions, which may be thought of as *generating* proof-objects.

3. Dynamic Assertions

Whilst the proofs as acts view does suggest that there might be a place for conditional assertions alongside categorical assertions, there remain obvious problems. It is fairly clear, for example, that this kind of approach is incompatible with accounts of assertion that require constitutive norms (such as the existence of proofs), which are supposed to govern the proprieties of assertions. On such accounts, assertions are taken to be an "all or nothing" affair. There are a number of accounts of assertion that require constitutive norms on the making of assertions, such as that one must make an assertion, "c", only in case one knows that "c", as held by Williamson [44]; or that one must make an assertion, "c", only in case it is true that "c", as held by Weiner [43]; or that one must make an assertion, "c", only in case a proof of "c" exists [8, e.g.]. These accounts all share the idea that categorical assertions are taken to be "industry standard". So, even if conditional assertions were allowed some sort of existence, the "all or nothing" status awarded to assertions means that conditional assertions would be treated as "empty". I will not discuss the merits, or otherwise, of these views (for this, see the excellent discussion in [19, 28, 29]).[16] Rather, I want to point to an alternative account of assertion that coheres much better with the approach to proofs and proving given above.

In [19], this is called the commitment view, where it is traced back to the work of Peirce [30] who suggests that 'to assert a proposition is to make oneself responsible for its truth' (p. 384).[17] The key distinction between this, and the constitutive norms approach is, as Macfarlane [19] puts it; '[...] while the constitutive rules approach looks at upstream norms - norms for *making* assertions - the commitment approach looks at downstream norms - the normative *effects* of making assertions'. This view is, perhaps, made most clear in the account of assertion games given by Robert Brandom [1, 4].[18] In [4], Brandom suggests that asserting that "c" is to undertake a commitment to defend "c" when challenged. So, the emphasis here is not on *pre*scribing the conditions under which it is permissible to make an assertion, but rather it is an account of what is *pro*scribed *after* an assertion has been made. So, plausibly the key norm on assertion is not a commitment to its truth, but rather a commitment to *defend* its truth. In this vein, Pagin [28] also draws attention to the relationship between assertions and promises, as discussed by Watson [42], where Watson notes their similarities. The key distinction between the two, according to Watson, is that the commitment involved is to something that is speaker-independent, which is just the defensibility, rather than the truth, of the assertion. So, again, on this view, the agent making an assertion is obliged to defend the assertion if challenged [42, p. 70]. That is to say, assertoric norms should not be understood to restrict what an agent ought to assert, instead they may be thought of as constraints on how agents respond to challenge in social and dialogical contexts. Furthermore, on this view, the norms on assertions have to do with a willingness to make an attempt to justify those assertions that an agent has brought into "the game of giving and asking for reasons" [2, p.57].[19] Importantly, then, unlike the constitutive norms approaches to assertion, on which some sort of grounds (presumably justificatory) for making an assertion are required of agents *prior* to making that assertion, the commitment approach requires only that an agent be prepared to make an attempt to justify the assertion *subsequent* to making that assertion.

It is in this sense, then, that making an assertion may be thought of as being akin to making a move in a game of reasons. This makes it available for scrutiny, so that, when asked, then agent should attempt to justify the assertion by way of providing some sort of reasons for it:[20]

> In asserting a claim one not only authorizes further assertions, but commits oneself to vindicate the original claim, showing that one is entitled to make it. Failure to defend one's entitlement to an assertion voids its social significance as inferential

> warrant for further assertions. It is only assertions one is entitled to make that can serve to entitle others to its inferential consequences. Endorsement is empty unless the commitment can be defended.[21] [1, p. 641]

So, on this view, making an assertion is primarily a matter of bringing that assertion into “play”. At this point, the assertion is subject to norms involving a commitment to its defense, to providing reasons for it, and allowing it to be “tested” through interaction with other reasons, counterexamples and so on. Assertions do not stand alone, on Brandom's [e.g. 4, p.167] view, rather, they stand in need of reasons, and it is in the context of language “games” that we ask for, and provide reasons for, our assertions.

This is to place assertions squarely where they belong: in a social setting that involves multiple agents.[22] Further still, what is important in this shift is that these are dialogical norms that are explanatorily prior to constitutive norms of truth or knowledge or justification.[23] If we think of this in relation to the distinction between conditional and categorical assertions, the commitment view can be thought of as putting conditional assertions first, whilst allowing that categorical assertions are *generated* by the dynamics of assertion games. It is not the case that a conditional assertion that never generates a categorical assertion is merely empty, it is just that assertion is not an all or nothing kind of affair. So, rather than thinking of the making of an assertion as expressing a fully formed propositional content, which may be thought of as true or false, we rather think of it as playing the statement as a kind of token in a game.[24] At this point, the statement may be treated hypothetically, and can be challenged and tested by other agents. It may, for example, be defended by the provision of reasons, and it may be contested by other reasons and counterexamples. It is also the case that, at some point during this process, the agents involved might agree that adequate justification has been provided for the original statement to be considered verified, or, indeed, that there is enough reason to think that it is false. At this point, it seems that we would be in a position to evaluate the statement as a kind of propositional content, in the usual way, but this occurs only after this interaction has occurred.

3.1 Conditionals in Interaction

Let us connect the above account of assertions with the discussion of hypothetical and actual proofs. It is clear that we can think of an initial assertion as a hypothetical proof. But, now, whilst the idea of a hypothetical proof is somewhat idiosyncratic, it is rendered more transparent if thought of as just a “play”, or “move”, in the assertion game. This may be transformed into an actual proof by providing a proof for each assumption, and at each stage of the argument providing reasons for the statement in response to “tests”. But, now, notice that this would suggest that the the act of proving is one which is intrinsically social, rather than just individual, and that, we now have an explanatory structure for this process, which is just that the agent must respond to any tests of the initial statement, where these tests are part and parcel of the process of providing a proof of the initial statement. So, the process of constructing a proof is just our “game of giving and asking for reasons”, which concerns the hypothetical, and that which is “in process”, whilst the product which is a construction is that which is made explicit (to use Brandom's [4] terminology) over the course of this process, to the point where the initial statement is justified. That is to say, we have provided an explanation of the BHK-style approach that takes a construction to be the construction of a justification, by means of a social account of the nature of assertions and justifications.

The above approach to assertions not only provides a way of thinking about the role of assertions socially, but it also explains *why* proofs should be taken as central to any such account. This is to situate the activity of proving squarely in the dialogical approach to logic.[25] In this setting, proofs are required to have clear explanatory value, which piggybacks upon the interaction between proponent and opponent:

Proponent's job is not only to “beat Opponent”; she also seeks to persuade Opponent of the truth of the conclusion, if he has granted the truth of the premises. In fact, the goal is not only to show that the conclusion follows from the premises, but also why it does; this corresponds to the idea that deductive arguments ought to have explanatory value. In this sense, Proponent and Opponent are cooperating in a common inquiry to establish what follows from the premises, and thus to further investigate the topic in question [25].

In our, more generalised setting, this is just because the utterance of an assertion brings with it a commitment to its justification, in response to requests for reasons, which is tantamount to asking for proofs of the assertion to be given. So, over the process of attempting to provide a proof, agents are making conditional assertions, which may yet become categorical. We can think of this, hypothetical register as a function of the kind discussed by Martino and Usberti. A conditional assertion is like a hypothetical proof insofar as it is a function that takes a categorical assertion (antecedent), and maps it into a categorical assertion (consequent).[26] Whenever no categorical assertion of the antecedent exists this function remains hypothetical insofar as there is nothing yet to map into the consequent. Whenever there is such an assertion, a categorical assertion of the consequent is made. We now have a simple explanation of this process, in terms of the distinction between the *activity* of proving, and the *object* produced (a valid proof), by way of the social, and dialogical, role of assertions.

4. Conclusions

The above account has provided a way of thinking through the dynamics of a social and dialogical approach to assertions, and, in doing so, we might also conjecture that a dual role for denial is required, alongside assertion. If, for example, we take the notion of the “game of giving and asking for reasons” seriously, then, we have a setup that involves both *assertions* and *tests*, which, in a simplified abstraction, we may think of as a interaction between two agents. If we take it that the making of an assertion brings with it a commitment to its defence, then we also require something to defend that assertion *against*. Above, I have mentioned tests, counterexamples, and so on. These *challenges* to the initial assertion may be characterised by means of denial, insofar as denial is understood to be a basic speech-act that is both distinct from, and non-interdefinable with, assertion.[27] That is to say, we can interpret the account given above as a kind of dialogue structure between the roles of prover and denier, where an assertion of a statement involves a commitment to its defence, and a denial of the statement involves a commitment to its challenge. As such, we can split Brandom's “game” into two parts: the *giving* of reasons on the part of “prover”, and the *asking* for reasons on the part of the “denier”. But, how this brief suggestion is to be developed, I leave for another paper.[28]

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Notes

1. See the excellent discussion in [17, ch.7] and [36].
2. This distinction does not cohere with distinction between the between a conditional assertion (if α, then I assert that β) and the assertion of a conditional (I assert that, if α, then β), which I do not discuss in the following, noting only that there may be interesting complications with assertions of these forms.
3. See the discussion in [38] for some suggestions.
4. This follows Edgington's [13] argument.
5. I will not consider model-theory in what follows, but it is worth saying that I do not think an approach making use of a partial interpretation function will deliver the correct results. I develop an *inferentialist* semantics for interactions that is consistent with the below account in [40].
6. The following is based upon the accounts in [10, 31, 41].
7. It is reductions that ensure that a derivation can be normalized, since successive reduction procedures ensure that any "roundabouts" in the derivation can be eliminated. Dummett, in [10, p. 254], calls the fact that every closed derivation in an intuitionistic entailment structure can be reduced to a canonical derivation, the "fundamental assumption". Whenever the introduction and elimination are in "harmony", this ensures that if the conclusion of an introduction rule is also the major premise of an elimination rule (at some point in a derivation), then it is possible to reduce that derivation to a derivation with the same premises and conclusion, without the "detour" through those steps.
8. As [34] puts it, according to Prawitz, '[a]n argument is valid if either it reduces to a non-logical justification of an atomic sentence, or it reduces to an argument whose last inference is an introduction inference and whose immediate subarguments are valid' (p. 7).
9. Furthermore, this objectively true notion of the proof of a statement is equivalent with its truth, according to Dummett [9].
10. This may still be considered anti-realist from the point of view of an agent's epistemic access to proofs, but proofs may be understood to be agency-independent insofar as they are independent from an agent's actual proving-activities. See the discussion in [7].
11. Note that in the standard Brouwer-Heyting-Kolmogorov semantics, categorical assertions are privieleged, though Kolmogorov's own interpretation in terms of "problems" bears some resemblance to the below account. For discussion, see [5].
12. This is discussed in [24].
13. The quotation has been altered slightly to reflect the fact that I am interested in conditional rather than knowability, but the point is theirs.
14. The notion of a hypothetical proof bears obvious similarity to Girard's [15] notion of paraproof.
15. The analogy Sundholm makes is that written proofs are like annotations for a game of chess, as opposed to proof-acts, which are like the game itself.
16. Additionally, it may well be the case that these views can be made compatible with the account proposed below in some way.
17. I won't discuss the theory of truth suggested by these pragmatic approaches here, but see [3] for an exposition of Brandom's approach.
18. This also follows Wittgenstein's suggestion that making an assertion is to make a move in a game [45, §22].
19. An excellent discussion of these issues can be found in [35].

20. See also the excellent discussion in [20].
21. On how this approach differs from Gricean accounts, see [1], where transmission models of communication are dispensed with in favour of an interactional model.
22. Pagin, in several places [27, 28, 29, e.g.], makes an argument to the effect that the social account of assertions does not, by itself, provide sufficient conditions on the nature of assertions, whilst he accepts that it may be the case that they provide necessary conditions. The discussion in [19] provides a useful rejoinder, though, in any case I do not think that this is an issue for the view espoused here. For example, the kinds of problems usually thought to face commitment approaches involve examples where assertions are made without explicitly making statements, through nonlinguistic signs, for example. I don't think that these are problematic for the account given here, since, it seems perfectly acceptable that one might ask for reasons for such signs, thereby clarifying them, in the same way as linguistic statements. A slightly different example given by Nunberg [26], and discussed in [19], is a waitress who states that "The ham sandwich left without paying". The waitress has made an assertion, though it does not seem correct to say that she has asserted that the ham sandwich left without paying. But, whilst this may seem *prima facie* problematic for a commitment view, I agree with Macfarlane [19], that, to the contrary, this view fares very well in this respect:
[...] if we wanted to settle, for example, whether Nunberg's waitress had asserted that a *sandwich* had left, or that a *person* who ordered a sandwich had left, we might ask with (if either) of these propositions she meant to commit herself to.
23. Shieh [37] puts this as follows:
To be taken as making an assertion, a speaker must acknowledge that the statement she is making is subject to assessment as correct or incorrect, by reference to what she would count as justifying it. (cited in [21])
24. See also [8] for a similar approach to the relationship between assertion and proving.
25. We should note, however, that the dialogical approach advanced here, and influenced by Brandom, takes dialogue to be a largely cooperative activity, in which we are interested in reasoning together, rather than "playing against each other" as in Lorenzen or Hintikka style games. See [20] for a similar distinction.
26. This does, of course, suggest that we do not have hard and fast criteria for determining which "reasons" will be taken to be sufficient to general categorical assertions, rather, we are allowing that this sufficiency may be decided only in the space of reasons, and by those agents involved. It may be argued that a more traditional realist approach to conditionals, therefore has a leg-up on the approach advanced here, since it is capable of providing a clear and objective account of the sufficient conditions under which conditionals may be truthfully made. However, it is precisely these supposedly objective conditions that get us into trouble with conditionals in the first place, and as Dummett put it, I think this is just 'ontological mythology' [11, p. 25].
27. That we require denials, alongside assertions, also coheres with the fact that on this approach to proofs, the negation of a statement can not be categorically asserted. Whilst this is a problem for a monological approach to proof, it is grist to the mill for the account offered here, in which we can naturally introduce denial alongside assertion.
28. Great thanks to Fabien Schang for incisive comments on an earlier version of this paper.

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The Football of Logic

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Abstract: An analogy is made between two rather different domains, namely: logic, and football (or soccer). Starting from a comparative table between the two activities, an alternative explanation of logic is given in terms of players, ball, goal, and the like. Our main thesis is that, just as the task of logic is preserving truth from premises to the conclusion, footballers strive to keep the ball as far as possible until the opposite goal. Assuming this analogy may help think about logic in the same way as in dialogical logic, but it should also present truth-values in an alternative sense of speech-acts occurring in a dialogue. The relativity of truth-values is focused by this way, thereby leading to an additional way of logical pluralism.
Keywords: assertion, denial, football, game, goal, logic, possession, speech-acts, truth-values, strategy, tactics.

1. Introduction: Logic of Football vs Football of Logic

Football (or the American "soccer") is not a serious thing, like logic. Or it should not be so, for those who see nothing in it but a distraction for the masses. Now such a game may be taken seriously, even if its usual depiction in terms of star salaries and hooliganism reduces this sport to a desperate feature of entertainment society. Game-theorists do know that games may be taken seriously. As a matter of fact, it can be said about football what the social constructivist Robert Cox claimed about states in the area of international relations, i.e., that they are what is done with them. What is or can be done with football, from a serious point of view? Our answer is: an explanatory model of logic, from a game-theoretical perspective. There are at least two reasons not to proceed in this way. On the one hand, such an approach to logic already exists under the heading of Lorenzen and Lorenz's dialogical logic [4], or Hintikka's Game-Theoretical Semantics [3]. So why bother with a provocative introduction of football in the very serious area of logic? On the other hand, philosophers who are reluctant to chiasms should note that there may be a "logic of football" in the sense of a set of recursive rules explaining how football proceeds, but not the converse. No serious sense should be given to an alleged "football of logic", accordingly, if such an expression assigns an explanatory role of football to logic but not the contrary one. And yet, the next sections are meant to show to what extent the game of football may throw some light on the game of logic. Or, at the least, that they can be compared with each other with no prominent role for either. A first logical reflection in football has been already made recently, in a paper where the activity of refereeing was compared to a peculiar game whose rules differ from the players' ones [5]. The present paper purports to push the logical line farther, accounting for inference rules of logic by means of the rules surrounding the football players themselves.

2. Question-Answer Game

The sort of game we want to focus now is a question-answer game of logic. The explanation borrows from a four-valued logic of acceptance and rejection $\mathbf{AR}_4$ [11]. Roughly speaking, it is a logic of information including a set of formulas and logical constants (negation, conjunction, disjunction, and a strong implication) interpreted into a domain of four structured values with Boolean elements. The reason why there are four logical values comes from the combination of single Boolean values depicted in the tradition of Belnap and Dunn's logic FDE [1]: replacing truth and falsity by data (or evidence) for or against a given proposition, it becomes possible to have inconsistent data where a statement is said to be both true or false. In the contrary case, it is said to be neither true nor false where no information is available. By doing so, this logical system endorses *rejectivism* – the linguistic or logical theory according to which negation is primarily a speech-act to be explained in terms of denial. By explaining the background of rejectivism and depicting the no-answer as a force indicator that plays the role of denial [12], it results in another view of negation as an opposite of affirmation. Thus, the two polar answers "yes" and "no" are the basic units of meaning conveyed by affording some information with a sentential content. There is no eternal Proposition descending from the Fregean "Third Realm" of thoughts [7], in such a perspective; that is, truth and falsity are nothing but commitments made by a speaker about what is accepted or strongly rejected in the speaker's belief set. There is no sufficient space to discuss at length about the epistemological stakes of rejectivism, in the following paper [10]. Instead, let us focus on the technical peculiarities of the four-valued system $\mathbf{AR}_4$.

One feature is the structured form of valuations $\mathbf{A}(p) = \langle a_1(p), a_2(p)\rangle$ in $\mathbf{AR}_4$, which consists in an ordered pair of answers $a_1(p)$ and $a_2(p)$ to whether there is evidence for or against a given sentence p, respectively. There are only two possible answers: yes (1), or no (0), $\mathbf{A}$ being a function mapping each answer $\mathbf{a}_i(p)$ onto 1 or 0. For sake of simplicity, the logical values of $\mathbf{AR}_4$ will be simplified in the form (x,y), where $x = \mathbf{a}_1(p)$ and $y = \mathbf{a}_2(p)$.

Another feature is the ensuing valuation function $\mathbf{A}(p)$, which applies from a given sentential content p to the set of logical values $\mathbf{4} = \{11,10,01,00\}$. The common point with FDE is the number of values and their informal interpretation: 11 means "true and false" ("both true and false" or B, in FDE), 10 means "true and not false" ("true only" or T, in FDE), 01 means "not true and false" ("false only" or F, in FDE), and 00 means "neither true nor false" ("None" or N, in FDE).

3. Strong Conditional

What of the core issue of this special volume: conditional, i.e. implication? The main advantage of the above valuation is to afford a stronger characterization of it; stronger, in the sense that some of the theorems where the material implication of classical logic essentially occurs do not hold in $\mathbf{AR}_4$. Our logical constant is a case of "defective" conditional [14], which means that the logical relation cannot be satisfied unless the antecedent is said true or, equivalently, accepted by the speaker. Such a feature cancels any form of the highly counterintuitive paradoxes of material implication, by virtue of which the whole relation holds once the antecedent is false. A proper definition of any logical constant amounts to specify the conditions under which the compound sentence can be told true or told false, given that these two truth-values are independent from each other in our rejectivist approach. Negation, conjunction and disjunction are defined in $\mathbf{AR}_4$ in the same way as in FDE. Conditional is defined in a non-standard way, however.

On the one hand, any conditional $p \rightarrow q$ is true, i.e. accepted by the speaker, if and only if both p and q are accepted. In symbols:

$\mathbf{a}_1(p \rightarrow q) = 1$ iff $\mathbf{a}_1(p) = \mathbf{a}_1(q) = 1$; $\mathbf{a}_1(p \rightarrow q) = 0$, otherwise.

On the other hand, $p \rightarrow q$ is rejected, i.e. strongly denied by the speaker, if and only if p is accepted and q is rejected. In symbols:

$\mathbf{a}_2(p \rightarrow q) = 1$ iff $\mathbf{a}_1(p) = 1$ and $\mathbf{a}_2(q) = 1$; $\mathbf{a}_2(p \rightarrow q) = 0$, otherwise.

Note that any no-answer does not mean a rejection but, rather, a mere denial of the speaker (for want of any sufficient evidence for or against the corresponding sentence). This means symbolically that $\mathbf{a}_1(p) = 0$ need not entail that $\mathbf{a}_2(p) = 1$. Note also that only the truth-condition of strong conditional $\rightarrow$ differs from that of classical or material implication $\supset$, which assumes that not affirming and rejecting are on a par:

$\mathbf{a}_1(p \supset q) = 1$ iff $\mathbf{a}_2(p) = 1$ or $\mathbf{a}_1(q) = 1$; $\mathbf{a}_1(p \supset q) = 0$, i.e. $\mathbf{a}_2(p \supset q) = 1$, otherwise.

A number of classical theorems about implication are not valid any more in $\mathbf{AR}_4$, symbolized by the non-truth-preserving relation $\models^*$:

(1) $p \rightarrow q \models^* \neg p \vee q$

(2) $p \rightarrow q \models^* \neg q \rightarrow \neg p$

(3) $\models^* p \rightarrow (q \rightarrow p)$

(4) $\models^* (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$

The failure of (1) is due to the failure of the paradoxes of material implication. The failure of (2) concerns the classical formulation of Modus Tollens (MT) and relates to the distinction between sentential negation and rejection in $\mathbf{AR}_4$. The failures of (3) and (4) rely on the defective import of strong conditional, to the effect that no conditional holds once its antecedent is not accepted.

At the same time, some fundamental properties of implication are preserved with strong conditional, with respect to the general relation of logical consequence:

(5) $\models p \rightarrow q$ iff $p \models q$

(6) $p \rightarrow q, p \models q$

(7) $p \rightarrow q, q \rightarrow r \models p \rightarrow r$

(5) is the Deduction Theorem, which means that strong conditional preserves truth once the antecedent is accepted; (6) is an expression of Modus Ponens (MP), just as (7) by contrast to the failure of the object language version of transitivity in (4). The main advantage of strong conditional is to avoid the paradoxes of material implication while avoiding any collapse with conjunction. Importantly, a central feature of conditional is lost in its lattice theoretical definition:

$$v(p \rightarrow q) = 1 \text{ iff } v(p) \leq v(q)$$

While this feature is mostly maintained as an essential property of implication in the literature (see Costa-Leite's paper, in the present issue), we consider the paradoxical behavior of material or classical implication as a by-product of it. At the same time, the failure of (MT) in (2) is taken to be a much more important result of $\mathbf{AR}_4$. If (MP) and (MT) are viewed as essential properties of implication, strong conditional must be adjusted to (MT) in order to be considered as a proper characterization of conditionality. This is done by redefining "Tollens" as a non-falsity preserving relation from q to p, rather than a truth-preserving relation from $\neg q$ to $\neg p$ [9]. In a nutshell, (MT) means the following: let a given speaker accept a conditional relation from p to q; if this speaker also assumes that the consequent q is not true in a given context, then it is not possible any more to make the whole conditional true in such a context, due to the defective import of our strong conditional; the least thing to do in order not to lose the game is to deny the antecedent, for such a conditional is made false exactly when p is true and q is false. This means that a player may not win without losing, either. The classical characterization of conditional results in a "all or nothing" or bivalent situation, and that is the reason why (MT) is depicted as it stands in classical logic. In our case, however, the gap between winning and losing a game is filled by weak denial, which corresponds to a case of draw. You can compare this case with dialogical logic, when the attacked player avoids asserting something in order not to make a fatal move and wants to extend the duration of the game as far as possible. The moral of this is that $\mathbf{AR}_4$ endorses a twofold view of inference: as a maximal, truth-preserving relation, on the one hand; as a minimal, non-falsity preserving relation, on the other hand. This echoes with some other words around many-valued inference.

We just described the meaning of strong conditional. Now let us consider its meaning from a special game-theoretical point of view that of football, together with the other logical constants of negation, conjunction, and disjunction.

4. Analogical Games

Comparison is not reason, admittedly. But comparison is clarification, and the present table proposes a couple of analogies between the lexical fields of football and logic. By the same way, it helps make sense of a pluralist view of logic including more than only one game strategy. Assuming a Tarskian view of logical as a theory of truth-preserving relation, let us see how rejectivism, negation and strong conditional may make sense in such a perspective.

Football	Logic
having the ball	truth
losing the ball	falsity
game fact	(provisory) conclusion
match (finite set of actions)	reasoning (finite set of inferences)
goal (successful shot)	valid inference
failed shot	invalid inference
own-goal	counter-inference
coach	reasoner
players	atomic sentences
tactics (moves between players)	molecular sentences
attack	assertion
defense	rejection
ball	interpretation function
ball preservation	logical consequence
referee	logic

Let us now translate the logical background into our paradigm of football. As depicted by the above table, football is a game in which a team strives to keep the ball in order to score a goal at least once more than the other team. The usual opposition between a Proponent and its Opponent in dialogical logic is taken for granted, hereby: the Proponent is any team performing a finite number of moves towards scoring a goal, while it is taken for granted that the Opponent is the opposite team within a kind of zero-sum game – whoever wins makes the other lose, although draw happens once both sides are not able to fulfill their requirement. The duration of a match is finite, and a number of tactics are displayed between the players of a same team in order to fulfill the expected task into a limited amount of time.

The values of truth and falsity are context-dependent and relative to the team possessing the ball; the latter plays the role of the function interpretation $\mathbf{A}(p)$, assigning truth with respect to the attacking team and falsity with respect to the defending team. It means that truth-values are like game situations, when the team having the ball tries to go forward until the opposite goal. A team has the ball and goes forward when $\mathbf{a}_1(p) = 1$; $\mathbf{a}_1(p) \neq 0$, otherwise. It does not have the ball and goes backward when $\mathbf{a}_2(p) = 1$, and $\mathbf{a}_2(p) \neq 0$ otherwise. Note that this analogy may also work with American football, where players have to progress through field yards until the adversary's opposite field. In other words, truth is what is to be preserved by a given team from the beginning to the end of an articulate action; on the contrary, falsity must be avoided by any team in order not to lose before the deadline. A game is like a sequence of related sets of moves, each of these leading either to a successful or unsuccessful action.

An intended advantage of four-valuedness is the asymmetry displayed between truth and falsity, on the one hand, acceptance and rejection on the other hand. Indeed, the team that does not have the ball may not be scored as it stands and is expected to defend its own goal so long as the ball is controlled by the opposite team. For instance, the team progressing with the ball attempts to score a goal just as a Proponent wants to go from premises to the conclusion without losing truth on the way of its whole thesis; and the defenders avoid being scored by defending well or intercepting the ball anew, just as the Opponent blocks the way from true premises to true conclusion by rejecting the truth of sentences or showing the falsity of the final sentence in a given inference.

In summary, our rejectivist reading of truth-values matches with the difference in football between defense and counterattack: a team may defend without counterattacking, just as an Opponent may undermine the truth-preserving course of the Proponent without asserting the falsity of the conclusion at hand. The latter may be merely untrue, for want of any available evidence, in a material sense, or deductive relation in a formal sense of truth.

5. Tactics

The well-known distinction between tactics and strategy is rendered hereby by a difference between molecular sentences and inferences. We pay attention now to the former, which has to do with the definition of logical constants. The meaning of a logical constant $\bullet$ is given by a way to preserve the ball between any teammates p and q, in our "football of logic". Progression on the field is viewed according to the ball owner. In terms of speech-acts, assertion is an offensive move performed by a speaker and expressed by a sentence, whether affirmative or negative; mere denial (no-answer to either truth or falsity) is a defensive move, whereas rejection or strong denial (yes-answer to falsity) is an attack on its own viewed from the opposite perspective as a counter-attack. Logical constants can then be defined as follows. Conjunction is like a one-two pass between a pair of teammates: the ball is preserved if the move is achieved; it is lost if the second player cannot keep the ball, or if the first player cannot do what he intended with the expected receiver. Disjunction, on the other hand, makes room for an alternative range of receivers to preserve the ball and progress with it. Negation is a reversed move, thereby taking the ball again in one way of another after defending or conversely. The most intriguing constant is that of our strong conditional, defined in a defective way in $\mathbf{AR}_4$. In our football of logic, its usual or classical truth-conditions cannot be maintained because of its too weak condition for keeping the ball: it is not enough for any two teammates to take the ball back or having it in order to give a genuine description of a "conditional" move of the form $p \rightarrow q$. If a player p has the ball, then the team is progressing forwards only if the ball is passed to the player q and in no other way. Conversely, the falsity-condition of conditional corresponds to a situation in which p's team is forced to go backwards after q lost the ball. The aforementioned distinction between strong and weak denial is a distinction between going backwards and being blocked by the opposite defense. If p has the ball and passes to q who loses it, the ball is recuperated by the other team; this is a counterpart of strong denial or rejection, i.e., $\mathbf{a}_2(p \rightarrow q) = 1$. If p has the ball and does not pass it to q, however, the ball may not be lost but the whole team does not progress forward; this is a counterpart of weak denial, i.e., $\mathbf{a}_1(p \rightarrow q) = 0$. The paradox of material implication corresponds to a situation in which the tactics realized by a team is at odds with what is naturally called by an implication. For when the antecedent is not affirmed, the situation is as if the team action has been aborted, thus borrowing from Quine's account on what conditional means informally [6, p. 21]:

> If the antecedent turns out to be false, our conditional affirmation is as if it had never been made.

After describing the moves made between players, we described what is meant by the logical constants of a logical system. As to the close relation between conditional and inference (or logical consequence), it corresponds to the essential difference between a means and an end: conditional is a means to go forwards by a necessary move from the player p to the player q; inference is the general aim of going forward until the opposite goal and more often than the opposite team. Conditional is nothing but one move to do so, in addition to negation, conjunction, and disjunction. The resemblance between conditional and consequence is due to their necessary forward movement from a point to another one: from a player to another one, with conditional; from a side to the other one, with inference.

Let us now consider inference in more details, especially with respect to the way of inferring through the central constant of the present issue: conditional.

6. Strategy

Strategy is meant as a selection of ordered tactics. When a team is endowed with a number of various players, a good coach is the one who takes the best decision with respect to their features and the expected result of winning the game. Now just as a linear logician treats propositions like limited and decreasing resources, a coach may consider that his players are not in position to achieve some offensive moves without undergoing fatal counterattacks. If so, then the team must reinforce its defense and rely upon some mistakes from the opposite team. Although attack is taken to be the best defense, the latter remains the best way not to lose when attackers are without sufficient resources to perform their expected function. Our defective view of conditional and the resulting minimal definition of inference have to do with the previous Italian football style of "catenaccio", which consists in letting the opponents attack without intending to take the ball again and by merely blocking their attempts. A good illustration of this spoiling strategy is systematic offside. Being offside is an irregular situation that cancels an attempted instance of inference from premise to conclusion; it does not enable to win the game of truth-telling throughout, as the case turns out to be with the looser truth-conditions of classical conditional. But it does not make lose one, either: a bivalent reading would present offside as a situation leading the sanctioned team to perform something like an own-goal, assuming that any move amounts either to scoring or being scored. As for the case in which the speaker rejects the truth of the consequent, the situation is more awkward since the player does not stand "off" the game by doing so: he can lose it, in case he then affirms the truth of the antecedent. Truth counts above all, admittedly: in football, the best way not to lose is to score more goals than the opposite team in order to win the match; but also, a defensive strategy can be viewed as a complementary strategy purporting not to be scored, that is, not to lose the match in the end. You can compare assertions with an offensive strategy, that of scoring goals. Now an assertion can be affirmative ($\mathbf{a}_1(p) = 1$), or negative ($\mathbf{a}_2(p) = 1$) as with strong denial or rejection. In the latter case, the search for falsity-claims might appear contrary to the logical purpose of preserving truth, just as it may seem irrational for a football team to play by scoring own goals. And yet, we can even imagine such queer games in which moves that help to win in one game are moves that make one lose in another version of this game.

The plurality of strategies may be equated with the realm of non-classical inferences in logic: when more than one pattern of inference is admitted beyond the mainstream Tarskian, truth-preserving relation from premises to conclusion. The latter matching with offensive strategy, a defensive one is like a non-falsity preserving relation. In the case of conditional, again, the two main properties (MP) and (MT) may help to illustrate this point. (MP) is nothing but a correct application of the conditional relation from *p* to *q*, given the successful tactics leading from a player *p* to his teammate *q*. With (MT), the defensive objective is not to go forward but, after being deprived of the ball, to avoid going backward by blocking the opposite move. This is made by blocking the false transition from a true antecedent *p* to a false consequent *q*. Given the valid inference from *p* to *q*, and given the non-truth of *q*, what the player *p* has to do is to not give the ball to *q*. Therefore, MT does not mean the success of an offensive strategy but, rather, the application of a least defensive strategy in order not to lose ground. Rather than winning a game, (MT) is to be viewed as a minimal rule that helps not to lose by playing logic in a reasoning including conditional.

7. Action

A final relevant comparison relates speech acts and actions in a game. In our approach implemented by the logical system $\mathbf{AR}_4$, truth and falsity are not abstract properties of sentences but actions made by them (see Trafford's paper, in the present volume). Taking sentences to be players, this means that no game action can be performed without a player to pass the ball, center towards a teammate, block the opposite offensive, counterattack after taking the ball back, and the like. Moreover, we said earlier that a common action can be viewed as an offensive or defensive phase of the game, depending upon the perspective from which the action is viewed. This is another reason not to ascribe truth-values to sentences as unique and abstract entities; these values are more like context-dependent data assigned to players, with limited resources and in a limited lapse of time. For this reason, the well-known distinction made by Dummett [2] between two senses of conditional hardly seems to make sense in our game-theoretical reading of logic.

According to Searle & Vanderveken, speech-acts are the primary vehicles of meaning; every speech-act is of the logical form F*p*, where F is a so-called "illocutionary force" and *p* a sentential content. It clearly appears that affirmations and denials, i.e. yes- and no-answers, are counterparts of F in the metalanguage of $\mathbf{AR}_4$. After assuming this background, an illocutionary account of conditional is proposed by Searle & Vanderveken in the vein of Dummett's analysis [13, p. 5]:

> It is essential to distinguish between a conditional speech act and a speech act whose propositional content is a conditional. In a conditional speech act expressed by a sentence of the form "If *p* then F(*q*)" the speech act expressed by "F(*q*)" is performed on condition *p*. Syntactically the "if" clause modifies the illocutionary force indicating device. This form is quite distinct from that of the speech act performed by an utterance of a sentence of the form "F(if *p* then *q*)" whose propositional content is conditional, for in this case an illocutionary act of force F is categorically performed. Thus, for example, in a bet on a conditional of the form (1) "I bet you five dollars that if a presidential candidate gets a majority of the electoral votes he will win" one either wins or loses five dollars depending on the truth or falsity of the conditional proposition (provided all the presuppositions hold). On the other hand, in a conditional bet of the form (2) "If Carter is the next Democratic candidate, I bet you five dollars that the Republicans will win",

there is a winner or a loser if Carter is the next Democratic candidate. The logical form of (2) is $p \rightarrow F(q)$. This conditional is not truth-functional, for from the fact that Carter does not run for the presidency, it does not follow that every speaker performs a conditional bet of the form (2).

The whole explanation relies on the view that truth-functionality has to do with truth and falsity, together with the assumption that these truth-values result in a winning or losing bet in every case. Since the second form of conditional may lead to neither, Searle & Vanderveken conclude from it that both conditionals do not mean the same. However, our account of conditional proposes a uniform explanation of how the speaker may win or lose the bet by putting a conditional expression. Unlike the above passage, rejection does not lead to non-truth-functionality when it means a mere no-answer. There are three truth-functionally different results for the speaker described by the above situation, in $\mathbf{AR}_4$: either the speaker of (2) asserts that Carter will be candidate and argues that the victory of Republicans will follow from it. In this case, the speaker is wrong and loses the bet since Carter did win the US elections in 1976. Or, he asserts the same antecedent while asserting that the Republicans will lose accordingly, in which case the speaker is right and wins the bet. Otherwise, the speaker may remain silent about whether Carter will be candidate or not; if so, then the whole conditional is merely rejected as well. This result is insightful inside our four-valued logic, however. Actually, there is nothing but a *pragmatic* difference between (1) and (2): in (1) the speaker commits in the truth of the antecedent, whereas in (2) (s)he does not commit in merely asserting a conditional relation between p and q. In our football of logic, it is as if a coach generally asserts that a move from p to q will turn out successful for the team. Why not doing it concretely, if so? Logically speaking, the point is that the actual assertion of p and q following the assumedly successful relation of the conditional $p \rightarrow q$ does not seem to differ from the conjunction $p \wedge q$: their success-conditions are the same, given that a rejection of the antecedent p does not make the whole relation true any longer in $\mathbf{AR}_4$. Nevertheless, there is no collapse of conditional to conjunction because their *falsity*-conditions differ: the speaker is wrong only if q does not follow from p once p is true, whereas nothing wrong is said once the antecedent p is not true.

Let us give two samples of logical inferences, the one being successful and the other unsuccessful.

The successful game fact is a valid inference is an instantiation of disjunctive syllogism, or Modus Tollendo Ponens (MTP): $p \vee q, \neg q \models p$.

Let us assume that the player p has the ball. Then p has the choice between two options in order for his own team to keep the ball: either keeping the ball for himself, or passing it to his teammate q. Now let us assume also that p does not want to pass the ball because of q being located in an inappropriate position (offside, or marked by an opponent very closely). Therefore, p decides to keep the ball in order for his team not to lose it.

The unsuccessful game fact is an invalidation of the current Modus Tollendo Tollens (MTT): $p \rightarrow q, \neg q \models \neg p$.

Any player p having the ball must pass it to his teammate q, if he wants his team not to lose it. Let us assume that this conditional requirement is supplemented with a circumstance in which q is losing the ball. Contrary to the standard interpretation of Modus Tollendo Tollens, the conclusion hereby is by no means that the player p should lose the ball either. For why on earth p should lose the ball under the pretext that q already lost it beforehand? A better view of our stronger reading of conditional is discussed at length in [9] and runs as follows. First of all, sentential negation and denial do not mean the same: a speaker may deny p without being in position to assert its falsity, $\neg p$. From a football game perspective of logic, such a difference is on a par with that between losing the ball and not having the ball, respectively. Correspondingly, the standard reading of Modus Tollens Tollens is blamed for deriving a meaningless conclusion to the effect that the player p must lose the ball if, by assuming that p must pass the ball to q in order for their team to keep it, q turns out to lose it. There is merely no logical relation between these distinctive data, actually. Alternatively, our stronger reading of implication helps to redefine MTT by replacing sentential negation by denial: the first premise is to the effect that *p* must pass the ball to *q*, in order for his team to keep the ball; now the second premise is *not* that *q* lost the ball but, rather, that *q* does merely not have the ball. For *q* would have received it otherwise, in accordance to the conditioned way of keeping the ball by passing the ball from *p* to *q*. From these two premises, it can thus be inferred that *p* does not have the ball either.

To sum up, conditional is never non-truth-functional in our dialogical explanation of how it means to make a consequent conditional upon an antecedent. At the same time, Dummett is still right in saying that there is a relevant difference between betting on a conditional proposition and making a bet conditional. For the two actions do not lead to the same result, and we can express such a difference by distinguishing affirmation and denial as two genuinely informative values.

8. Conclusion: Towards a Pluralist Football?

The present paper has attempted to use the background of football as a game-theoretical framework for logical reasoning. A normal stance should consist in following a contrary process, i.e., making use of logic as a proper explanatory model of football rather than the converse. However, any analogy between the two activities does not impose any ordering explanatory relation between them. In other words, truth in logic is like ball possession in football and conversely; scoring is like preserving truth until the conclusion, given that the opposite goal represents a successful conclusion; the various ways for footballers of keeping or taking the ball back are like the various ways of having evidence for the truth of propositions; and the like.

Let us close this sketchy approach to a football of logic by reopening the question of logical pluralism, and its significance in the area of sports like football. What should it mean to accept more than one consequence relation in a sport game? Card games already give some such exemplifications, when the normal rules of a card game are deeply modified and can even go on inverting the usual purpose – think about cases where the worst player turns into the best one, thereby finding strategies to lose from the standard game in order to win the game from the non-standard one. In football like in science, success is the main target in that telling the truth is as much central for a normal logician as scoring for footballers. Such alternative ways of playing logic have been illustrated elsewhere, including the case of Indian logics and their emphasis on non-standard targets like peaceful agreement with tolerance in dialogues [8]. Zero-sum games are so conspicuous that it may seem difficult to imagine any dialogical explanation of logic in such a vein. However, our relativisation of what logic means through analogy is in position to change our mind in two ways: about what both logic and football are used for. The present paper is nothing but an introductory attempt to do so, by means of a comparative analysis of what logic and sports have fundamentally in common: being a game, without which these activities cannot make sense in a community of players.

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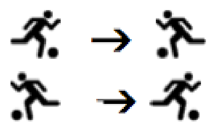
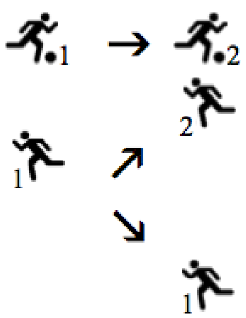
Appendix: The Meaning of Logical Constants in a Football of Logic.

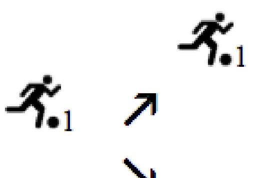
For every sentence φ (p, q, $p \bullet q$, etc.) of the form $\mathbf{A}(\varphi) = \langle \mathbf{a}_1(\varphi), \mathbf{a}_2(\varphi)\rangle$, the logical constants $\bullet = \{\neg,\wedge,\vee,\rightarrow\}$ of $\mathbf{AR}_4$ are defined by their truth- and falsity-conditions. In the left part of the definitions, the left and right items of ordered pairs $\langle x,y\rangle$ express the truth- and falsity-conditions of the corresponding compound sentence $p \bullet q$; respectively. In the right part, the football logos are twofold: players having the ball and attacking (going forward, from left to right) symbolize truth; players deprived from the ball and defending (going backward, from right to left) symbolize falsity. The top schemes stand for truth-conditions, and the bottom schemes for falsity-conditions.

Negation
$\mathbf{A}(\neg p) = \langle \mathbf{a}_2(p), \mathbf{a}_1(p)\rangle$



Conjunction
$\mathbf{A}(p \wedge q) = \langle \mathbf{a}_1(p) \cap \mathbf{a}_1(q), \mathbf{a}_2(p) \cup \mathbf{a}_2(q)\rangle$





Disjunction

$\mathbf{A}(p \vee q) = \langle \mathbf{a}_1(p) \cup \mathbf{a}_1(q),\ \mathbf{a}_2(\mathrm{p}) \cap \mathbf{a}_2(q) \rangle$

Conditional

$\mathbf{A}(p \rightarrow q) = \langle \mathbf{a}_1(p) \cap \mathbf{a}_1(q),\ \mathbf{a}_1(\mathrm{p}) \cap \mathbf{a}_2(q) \rangle$